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ELEMENTARY TRIGONOMETRY

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ELEMENTARY TRIGONOMETRY

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PREFACE.

I HAVE attempted in this work to explain and illustrate the principles of that portion of Plane Trigonometry which precedes De Moivre's Theorem. The method of explanation is similar to that adopted in my ELEMENTARY ALGEBRA. The examples, progressive and easy, have been selected chiefly from College and University Examination Papers, but I am indebted for many to the works of several German writers, especially those of Dienger, Meyer, Weiss, and Wiegand. I have carried on the subject somewhat beyond the limits set by the Regulations for the Examination of Candidates for Honours in the Previous Examination for two reasons: first, because I hope to see those limits extended, secondly, that my work may be more useful to those who are reading the subject in Schools and to Candidates in the Local Examinations.

J. H. SMITH.

CAMBRIDGE,
Feb. 18, 1870.

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ELEMENTARY TRIGONOMETRY.

CHAPTER I.

ON THE APPLICATION OF ALGEBRA TO CERTAIN GEOMETRICAL PROPOSITIONS.

1. TRIGONOMETRY was originally, as the name imports, the science which furnished methods for determining the magnitude of the sides and angles of triangles, but it has been extended to the treatment of all theorems involving the consideration of angular magnitudes.

2. By way of introduction to the subject we have to treat in this chapter, first, of the representation of lines by Algebraic Symbols, and, secondly, of the relation existing between the circumference of a circle and its diameter.

3. To measure a line AB we fix upon some line as a standard of linear measurement: then if AB contains the standard line p times, p is called the *measure* of AB, and the magnitude of AB is represented algebraically by the symbol p.

Since the standard contains itself once, its measure is *unity*, and it will be represented by 1.

4. Two lines are said to be *commensurable* when each contains a line taken as the standard of measurement (or, as it is commonly called, the unit of length) an exact number of times.

5. If the measures of two lines AB, CD be p and q respectively, the ratio $AB : CD$ is represented by the fraction $\frac{p}{q}$.

6. Magnitudes of things cannot properly be made subject to the rules and operations of Algebra, as these rules and operations have only been proved for algebraical symbols. We must therefore find some algebraical representative for any magnitude before we subject it to algebraical operations: such a representative is the measure of that magnitude with the proper sign prefixed according to rules to be given hereafter.

7. In the application of Algebra to Geometry it is the practice of most writers to use the geometrical representative of a magnitude where the algebraical representative ought to be employed. Thus suppose p and q to be the measures of two lines AB, CD, we often find the fraction $\frac{AB}{CD}$ where we ought in strictness to find the fraction $\frac{p}{q}$. This loose method of notation is, however, sometimes less cumbersome, and we shall therefore retain it at the risk of a slight want of clearness.

EXAMPLES.—I.

(1) If the unit of length be an inch, by what number will 4 feet 6 inches be represented?

(2) If 7 inches be taken as the unit of length, by what number will 15 feet 2 inches be represented?

(3) If 192 square inches be represented by the number 12, what is the unit of linear measurement?

(4) If 1000 square inches be represented by the number 40, what is the unit of linear measurement?

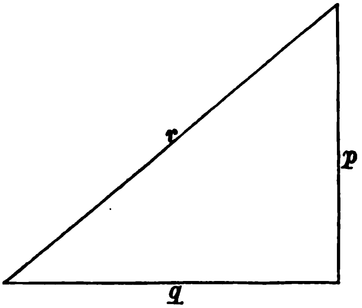
(5) If 216 cubic inches be represented by the number 8, what is the unit of linear measurement?

(6) If 2000 cubic inches be represented by the number 16, what is the unit of linear measurement?

(7) If a yards be the unit of length what is the measure of b feet?

(8) If the measure of m yards be c, what is the measure of n feet?

8. Now suppose the measures of the sides of a right-angled triangle to be p, q, r respectively, the right angle being subtended by that side whose measure is r.



Then since the geometrical property of such a triangle, established by Euclid I. 47, may be extended to the case in which the sides are represented by *numbers* or symbols standing for numbers,

$$p^2 + q^2 = r^2.$$

If any two of the numerical quantities involved in this equation are given, we can determine the third.

For example, if $r = 5$ and $q = 3$,

$$p^2 + 9 = 25,$$
$$\therefore\ p^2 = 16,$$
$$\therefore\ p = 4.$$

EXAMPLES.—II.

(1) The hypothenuse being 51 yards, and one of the sides containing the right angle 24 yards, find the other side.

(2) The sides containing the right angle being 8 feet and 6 feet, *find the hypothenuse.*

(3) A rectangular field measures 225 yards in length, and 120 yards in breadth; what will be the length of a diagonal path across it?

(4) A rectangular field is 300 yards long and 200 yards broad; find the distance from corner to corner.

(5) A rectangular plantation, whose width is 88 yards, contains $2\frac{1}{2}$ acres; find the distance from corner to corner across the plantation.

(6) The sides of a right-angled triangle are in Arithmetical Progression and the hypothenuse is 20 feet; find the other sides.

(7) The sides of a right-angled triangle are in Arithmetical Progression; shew that they are proportional to 3, 4, 5.

(8) A ladder, whose foot rests in a given position, just reaches a window on one side of a street, and when turned about its foot, just reaches a window on the other side. If the two positions of the ladder be at right angles to each other, and the heights of the windows be 36 and 27 feet respectively, find the width of the street and the length of the ladder.

(9) In a right-angled isosceles triangle the hypothenuse is 12 feet, find the length of each of the other sides.

(10) What is the length of the diagonal of a square, whose side is 5 inches?

(11) The area of a square is 390625 square feet; what is the diagonal?

(12) Each side of an equilateral triangle is 13; find the length of the perpendicular dropped from one of the angles on the opposite side.

(13) If ABC be an equilateral triangle and the length of AD, *a perpendicular on* BC, be 15; find the length of AB.

(14) The radius of a circle is 37 inches; a chord is drawn in the circle: if the length of this chord be 70 inches, find its distance from the centre.

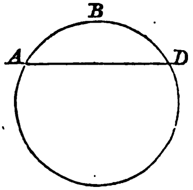
(15) The distance of a chord in a circle from the centre is 180 inches; the diameter of the circle is 362 inches: find the length of the chord.

(16) The length of a chord in a circle is 150 feet, and its distance from the centre is 308 feet; find the diameter of the circle.

9. We now proceed to treat of the ratio of the circumference of a circle to the diameter of that circle.

10. It is evident that a straight line can be compared as to its length with a circular arc, and that consequently the ratio between such lines can be represented in the form of a fraction.

11. We must assume as an axiom that an arc is greater than the chord subtending it: that is, if *ABD* be part of the circumference of a circle cut off by the straight line *AD*, the length of *ABD* is greater than the length of *AD*.



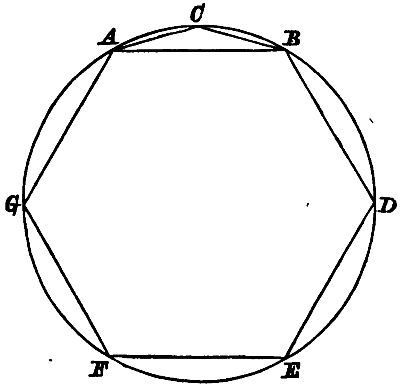
12. A figure enclosed by any number of straight lines is called a *polygon*.

13. A *regular* polygon is one in which all the sides and angles are equal.

14. The *perimeter* of a polygon is the sum of the sides. Hence if *AB* be one of the sides of a regular polygon of n sides the perimeter of the polygon will be $n . AB$.

15. The circumference of a circle is greater than the perimeter *of any polygon which* can be inscribed in the circle, but as the

number of sides of such a polygon is increased the perimeter of the polygon approaches nearer to the circumference of the circle, as will appear from the following illustration.



Let AB be the side of a regular hexagon $ABDEFG$ inscribed in a circle.

Then AB is equal to the radius of the circle. Eucl. iv. 15.

Now the arc ACB is greater than AB, and the circumference of the circle is therefore larger than the perimeter of the hexagon.

Hence

the circumference is greater than six times the radius,

and greater than *three times the diameter.*

Now suppose C to be the middle point of the arc AB.

Join AC, CB.

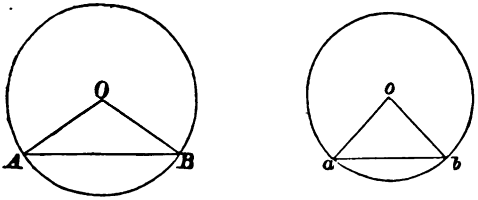
These will be sides of a duodecagon or regular figure of 12 sides inscribed in the circle.

Now AC, CB are together *greater than* AB: but AC, CB are together *less than the arc* ACB.

Hence the perimeter of the duodecagon will be less than the circumference of the circle, but will approximate more nearly to the circumference of the circle than the perimeter of the hexagon.

So the larger the number of sides of a polygon inscribed in a circle, the more nearly does the perimeter of the polygon approach to the circumference of the circle; and when the number of sides is infinitely large, the perimeter of the polygon will become ultimately equal to the circumference of the circle.

16. *To shew that the ratio of the circumference of a circle to the radius is the same for all circles.*



Let O and o be the centres of two circles.

Let AB, ab be sides of regular polygons of n sides inscribed in the circles, P, p the perimeters of the polygons, and C, c the circumferences of the circles.

Then OAB, oab are similar triangles.

$$\begin{aligned} \therefore\ OA : oa &:: AB : ab \\ &:: n.AB : n.ab \\ &:: P : p. \end{aligned}$$

Now when n is very large the perimeters of the polygons may be regarded as equal to the circumferences of the circles;

$$\therefore\ OA : oa :: C : c.$$

Hence it follows that the circumference of a circle varies as the radius of the circle.

17. Since the circumference varies as the radius, the ratio $\dfrac{\text{circumference}}{\text{radius}}$ is the same for all circles, and therefore the ratio $\dfrac{\text{circumference}}{\text{diameter}}$ is the same for all circles.

18. Def. The ratio $\dfrac{\text{circumference}}{\text{diameter}}$ is denoted by the symbol π.

19. The value of this numerical quantity π cannot be determined exactly, but it has been *approximately* determined by various methods.

If we take a piece of string which will exactly go round a penny, and another piece which will exactly stretch across the diameter of the penny: if we then set off along a straight line seven lengths of the first string, and on another straight line by the side of the first we set off twenty-two lengths of the second string, we shall find that the two lines are very nearly equal. Hence 22 diameters are nearly equal to 7 circumferences, that is the ratio $\dfrac{\text{circumference}}{\text{diameter}} = \dfrac{22}{7}$ nearly, or in other words the fraction $\dfrac{22}{7}$ is a rough approximation to the value of π.

The fraction $\dfrac{355}{113}$ gives a closer approximation.

The accurate value of the ratio to 5 places of decimals is $3{\cdot}14159$.

20. Suppose we call the radius of a circle r: then the diameter $= 2r$.

Now
$$\frac{\text{circumference}}{\text{diameter}} = \pi;$$
$$\therefore \frac{\text{circumference}}{2r} = \pi;$$
$$\therefore \text{circumference} = 2\pi r.$$

Hence arc of semicircle $= \pi r$,

and arc of quadrant $= \frac{\pi r}{2}$.

EXAMPLES.—III.

In the following examples the value of π may be taken as $\frac{22}{7}$.

(1) The diameter of a circle is 5 feet, what is its circumference?

(2) The circumference of a circle is 542 ft. 6 in., what is its radius?

(3) The driving-wheel of a locomotive-engine of diameter 6 feet makes 2 revolutions in a second: find approximately the number of miles per hour at which the train is going.

(4) Supposing the earth to be a perfect sphere whose circumference is 25000 miles, what is its diameter?

(5) The diameter of the Sun is 883220 miles, what is its circumference?

(6) The circumference of the moon is 6850 miles, what is its radius?

(7) Find the length of an arc which is $\frac{1}{12}$ of the whole circumference, if the radius is 12 ft. 6 in.

(8) Find the length of an arc which is $\frac{5}{7}$ of the whole circumference, if the diameter is 21 feet.

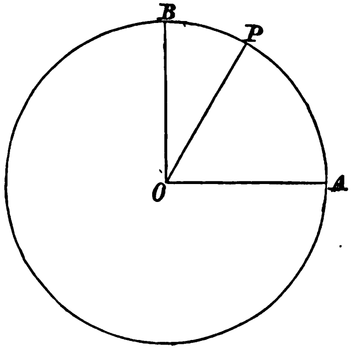
(9) The circumference of a circle is 150 feet, what is the side of a square inscribed in it?

(10) The circumference of a circle is 200 feet, what is the side of a square inscribed in it?

(11) A water-wheel, whose diameter is 12 feet, makes 30 revolutions per minute. Find approximately the number of miles per hour traversed by a point on the circumference of the wheel.

(12) A mill-sail, whose length is 21 feet, makes 15 revolutions per minute. How many miles per hour does the end of the sail traverse?

21. *The angle subtended at the centre of a circle by an arc equal to the radius of the circle is the same for all circles.*



Let O be the centre of a circle, whose radius is r;

AB the arc of a quadrant, and therefore AOB a right angle;

AP an arc equal to the radius AO.

Then, $AP = r$ and $AB = \frac{\pi r}{2}$. (Art. 20.)

Now, by Euc. VI. 33,

$$\frac{\text{angle } AOP}{\text{angle } AOB} = \frac{\text{arc } AP}{\text{arc } AB},$$

or,

$$\frac{\text{angle } AOP}{\text{a right angle}} = \frac{r}{\frac{\pi r}{2}} = \frac{2r}{\pi r} = \frac{2}{\pi}.$$

Hence
$$\text{angle } AOP = \frac{2 \text{ right angles}}{\pi}.$$

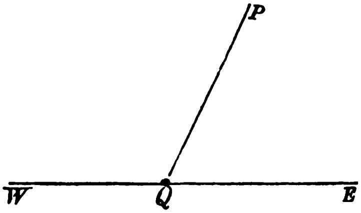
Thus the magnitude of the angle AOP is independent of r and is therefore the same for all circles.

CHAPTER II.

ON THE MEASUREMENT OF ANGLES.

22. Euclid defines a plane rectilineal angle as the inclination of two straight lines to each other, which meet, but are not in the same straight line. Hence an angle in Geometry must be less than two right angles.

In Trigonometry the term *angle* is used in a more extended sense, the magnitude of angles in this science being unlimited.



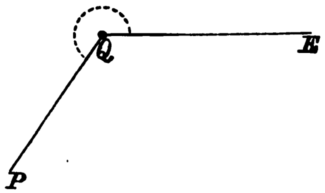
23. An angle in Trigonometry is defined in the following manner.

Let WQE be a fixed straight line, and QP a line which revolves about the fixed point Q, and which at first coincides with QE.

Then when QP is in the position represented in the figure, we say *that it has described the angle PQE.*

The advantage of this definition is that it enables us to consider angles not only greater than two right angles, but greater than four right angles, viz. such as are described by the revolving line when it makes more than one complete revolution.

24. In speaking of a trigonometrical angle we must take into account the position from which the line that has described the angle started.



Suppose for instance that QP, starting from the position QE and revolving in a direction contrary to that in which the hands of a watch revolve, has come into the position indicated in the figure. It has then described an angle EQP greater than two right angles.

25. The magnitudes of angles are represented by numbers expressing how many times the given angles contain a certain angle fixed upon as the unit of angular measure.

When we speak of an angle θ we mean an angle which contains the unit of angular measurement θ times.

26. There are three modes of measuring angles, called

I. The Sexagesimal or English method,

II. The Centesimal or French method,

III. The Circular Measure,

which we now proceed to describe in order.

I. *The Sexagesimal Method.*

27. In this method we suppose a right angle to be divided into 90 equal parts, each of which parts is called a *degree*, each degree to be divided into 60 equal parts, each of which is called a *minute*, and each minute to be divided into 60 equal parts, each of which is called a *second*. Then the magnitude of an angle is expressed by the number of degrees, minutes and seconds, which it contains. Degrees, minutes and seconds are marked respectively by the symbols $^\circ$, $'$, $''$: thus, to represent 14 degrees, 9 minutes, 37·45 seconds, we write $14^\circ.9'.37''\cdot45$.

28. We can express the measure of an angle (expressed in degrees, minutes and seconds) in degrees and decimal parts of a degree by the following process.

Let the given angle be $39^\circ.5'.33''$,

$$5'.33'' = 333''.$$

Then since $$1^\circ = 3600'',$$

$$333'' = \frac{333}{3600} \text{ of a degree}$$

$$= \cdot 0925 \text{ of a degree};$$

$$\therefore\ 39^\circ.5'.33'' = 39\cdot 0925 \text{ degrees.}$$

Examples.—IV.

Express as the decimal of a degree the following angles.

(1) $24^\circ.\ 16'.\ 5''$,

(2) $37^\circ.\ 2'.\ 43''$,

(3) $175^\circ.\ 0'.\ 14''$,

(4) $5'.\ 28''$,

(5) $375^\circ.\ 4'$,

(6) $78^\circ.\ 12'.\ 4''$.

II. *The Centesimal Method.*

29. In this method we suppose a right angle to be divided into 100 equal parts, each of which parts is called a *grade*, each grade to be divided into 100 equal parts, each of which is called a *minute*, and each minute to be divided into 100 equal parts, each of which is called a *second*. Then the magnitude of an angle is expressed by the number of grades, minutes and seconds, which it contains. Grades, minutes and seconds are marked respectively by the symbols ^{g}, $^{`}$, $^{``}$: thus, to represent 35 grades, 56 minutes, 84·53 seconds, we write $35^g.\ 56^{`}.\ 84^{``}\cdot 53$.

The advantage of this method is that we can write down the minutes and seconds as the decimal of a grade by inspection.

Thus, if the given angle be $14^{g}.\ 19^{\backprime}.\ 57^{\backprime\backprime}$,

$$\text{since } 19^{\backprime} = \frac{19}{100} \text{ of a grade} = \cdot 19 \text{ grades,}$$

$$\text{and } 57^{\backprime\backprime} = \frac{57}{10000} \text{ of a grade} = \cdot 0057 \text{ grades,}$$

$$14^{g}.\ 19^{\backprime}.\ 57^{\backprime\backprime} = 14\cdot 1957 \text{ grades.}$$

30. If the number expressing the minutes or seconds has only one significant digit, we must prefix a cipher to occupy the place of tens before we write down the minutes and seconds as the decimal of a grade.

Thus $25^{g}.\ 9^{\backprime}.\ 54^{\backprime\backprime}. = 25^{g}.\ 09^{\backprime}.\ 54^{\backprime\backprime}$

$= 25\cdot 0954$ grades,

and $36^{g}.\ 8^{\backprime}.\ 4^{\backprime\backprime}. = 36^{g}.\ 08^{\backprime}.\ 04^{\backprime\backprime}$

$= 36\cdot 0804$ grades.

EXAMPLES.—V.

Express as decimals of a grade the following angles:

(1) $25^{g}.\ 14^{\backprime}.\ 25^{\backprime\backprime}$,

(2) $38^{g}.\ 4^{\backprime}.\ 15^{\backprime\backprime}$,

(3) $214^{g}.\ 3^{\backprime}.\ 7^{\backprime\backprime}$,

(4) $15^{\backprime}.\ 7^{\backprime\backprime}\cdot 45$,

(5) $425^{g}.\ 13^{\backprime}.\ 5^{\backprime\backprime}\cdot 54$,

(6) $2^{g}.\ 2^{\backprime}.\ 2^{\backprime\backprime}\cdot 22$.

31. The Centesimal Method was introduced by the French Mathematicians in the 18th century. The advantages that would have been obtained by its use were not considered sufficient to counterbalance the enormous labour which must have been spent on the re-arrangement of the Mathematical Tables then in use.

III. *The Circular Measure.*

32. In this method, which is chiefly used in the higher branches of Mathematics, the unit of angular measurement may be described as

(1) The angle subtended at the centre of a circle by an arc equal to the radius of the circle,

or, which is the same thing, as we proved in Art. 21, as

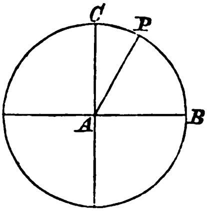
(2) The angle whose magnitude is the πth part of two right angles.

33. It is important that the beginner should have a clear conception of the size of this angle, and this he will best obtain by considering it relatively to the magnitude of that angular unit which we call a degree.

Now the unit of circular measure

$$= \frac{\text{two right angles}}{\pi} = \frac{180^\circ}{3{\cdot}14159} = 57^\circ{\cdot}2958 \text{ nearly.}$$

Now if BC be the quadrant of a circle, and if we suppose the arc BC to be divided into 90 equal parts, the right angle BAC will be divided by the radii which pass through these points into 90 equal angles, each of which is called a degree.



A radius AP meeting the arc at a certain point between the 57th and 58th divisions, reckoned from B, will make with AB an angle equal in magnitude to the unit of circular measure.

Hence an angle whose circular measure is 2 contains rather more than 114 degrees, and one whose circular measure is 3 contains about 172 degrees, or rather less than two right angles.

34. Again, since the unit of circular measure $= \frac{2 \text{ right angles}}{\pi}$,

π times the unit of circular measure = 2 right angles.

Hence

an angle whose circular measure is π is equal to 2 right angles,

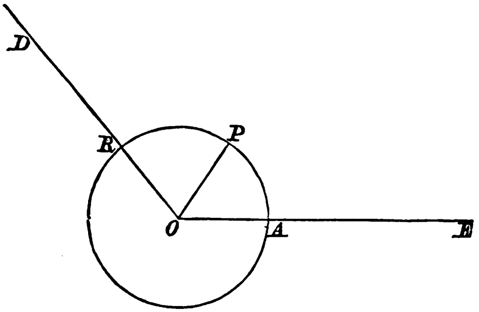
.... $\frac{\pi}{2}$ .... a right angle,

.... 2π .... 4 right angles.

35. *To shew that the circular measure of an angle is equal to a fraction which has for its numerator the arc subtended by that angle at the centre of any circle, and for its denominator the radius of that circle.*

Let EOD be any angle.

About O as centre and with any radius, describe a circle cutting OE in A, and OD in R.



Make angle AOP equal to the unit of circular measure.

Then arc AP = radius AO (Art. 32).

Now, by Euc. VI. 33,

$$\frac{\text{angle } AOR}{\text{angle } AOP} = \frac{AR}{AP};$$

$$\therefore \text{ angle } AOR = \frac{AR}{AP} \cdot \text{angle } AOP$$

$$= \frac{AR}{AO} \cdot \text{angle } AOP$$

$$= \frac{\text{arc}}{\text{radius}} \cdot \text{unit of circular measure};$$

$\therefore$ circular measure of angle AOR

$$= \frac{\text{arc}}{\text{radius}}.$$

36. The units in the three systems when expressed in terms of one common standard, two right angles, stand thus:

$$\text{the unit in the Sexagesimal Method} = \frac{1}{180} \text{ of two right angles,}$$

$$\text{the unit in the Centesimal Method} = \frac{1}{200} \text{ of two right angles,}$$

$$\text{the unit in the Circular Method} = \frac{1}{\pi} \text{ of two right angles.}$$

37. It is not usual to assign any distinguishing mark to angles estimated by the Third Method, but for the purpose of stating the relation between the three units in a clear and concise form, we shall use the symbol 1^c to express the unit of circular measure.

Then we express the relation between the units thus:

$$1^\circ : 1^g : 1^c = \frac{1}{180} : \frac{1}{200} : \frac{1}{\pi}.$$

CHAPTER III.

ON THE METHOD OF CONVERTING THE MEASURES OF ANGLES FROM ONE TO ANOTHER SYSTEM OF MEASUREMENT.

38. WE proceed to explain the process for converting the measures of angles from each of the three systems of measurement described in Chap. II. to the other two.

39. To convert the measure of an angle expressed in degrees to the corresponding measure in grades.

Let the given angle contain D degrees.

$$1 \text{ degree} = \frac{1}{90} \text{ of a right angle};$$

$$\begin{aligned} \therefore D \text{ degrees} &= \frac{D}{90} \text{ of a right angle} \\ &= \frac{D}{90} \text{ of } 100 \text{ grades} \\ &= \frac{100D}{90} \text{ grades} \\ &= \frac{10D}{9} \text{ grades.} \end{aligned}$$

Hence we obtain the following rule: If an angle be expressed in degrees, multiply the measure in degrees by 10, divide the result by 9, and you obtain the measure of the angle in grades.

Ex. How many grades are contained in the angle $24^\circ.\,51'.\,45''$?

$$24^\circ.\,51'.\,45'' = 24{\cdot}8625 \text{ degrees}$$

$$\begin{array}{r|l} & \;10 \\ \hline 9 & 248{\cdot}625 \\ \hline \text{grades} & \;27{\cdot}625 \end{array}$$

$\therefore$ the angle contains $27^{g}.\,62^{\backprime}.\,50^{\backprime\backprime}$.

EXAMPLES.—VI.

Find the number of grades, minutes and seconds in the following angles:

(1) $27^\circ.15'.46''$.   (2) $157^\circ.4'.9''$.
(3) $24.18''$.   (4) $19^\circ.0'.18''$.
(5) $143^\circ.9'$.   (6) 28°.
(7) $10^\circ.25'.48''$.   (8) $27^\circ.38'.12''$.
(9) $300^\circ.15'.58''$.   (10) $422^\circ.7'.22''$.

40. To convert the measure of an angle expressed in grades to the corresponding measure in degrees.

Let the given angle contain G grades.

$$1 \text{ grade} = \frac{1}{100} \text{ of a right angle};$$

$$\therefore G \text{ grades} = \frac{G}{100} \text{ of a right angle}$$
$$= \frac{G}{100} \text{ of 90 degrees}$$
$$= \frac{90G}{100} \text{ degrees}$$
$$= \frac{9G}{10} \text{ degrees.}$$

Hence we obtain the following rule: If an angle be expressed in grades, multiply the measure in grades by 9, divide the result by 10, and you obtain the measure of the angle in degrees.

Ex. How many degrees are contained in the angle $42^g.34^{\backprime}.56^{\backprime\backprime}$?

$$42^g.34^{\backprime}.56^{\backprime\backprime} = 42\cdot3456 \text{ grades}$$

$$\begin{array}{rl} & 9 \\ 10 & \underline{\big|\, 381\cdot1104} \\ \text{degrees} & 38\cdot11104 \\ & \underline{\quad\quad\;\; 60} \\ \text{minutes} & 6\cdot66240 \\ & \underline{\quad\quad\;\; 60} \\ \text{seconds} & 39\cdot74400 \end{array}$$

[illegible]he angle contains $38^\circ.\,6'.\,39''\cdot744$.

EXAMPLES.—VII.

Find the number of degrees, minutes, and seconds in the following angles:

(1) $19^{\text{g}}.\,45^{\backprime}.\,95^{\backprime\backprime}$.

(2) $124^{\text{g}}.\,5^{\backprime}.\,8^{\backprime\backprime}$.

(3) $29^{\text{g}}.\,75^{\backprime}$.

(4) $15^{\text{g}}.\,0^{\backprime}.\,15^{\backprime\backprime}$.

(5) $154^{\text{g}}.\,7^{\backprime}.\,24^{\backprime\backprime}$.

(6) 43^{g}.

(7) $38^{\text{g}}.\,71^{\backprime}.\,20^{\backprime\backprime}\cdot 3$.

(8) $50^{\text{g}}.\,76^{\backprime}.\,94^{\backprime\backprime}\cdot 3$.

(9) $170^{\text{g}}.\,63^{\backprime}.\,27^{\backprime\backprime}$.

(10) $324^{\text{g}}.\,13^{\backprime}.\,88^{\backprime\backprime}\cdot 7$.

41. If the number of degrees in an angle be given, to find its circular measure.

Let the given angle contain D degrees.

$$1^\circ = \frac{1}{180} \text{ of two right angles}$$

$$= \frac{1}{180} \text{ of } \pi \text{ units of circular measure}$$

$$= \frac{\pi}{180} \text{ units of circular measure;}$$

$$\therefore\ D^\circ = \frac{D\pi}{180} \text{ units of circular measure.}$$

Hence we obtain the following rule:

If an angle be expressed in degrees, multiply the measure in degrees by π, divide the result by 180, and you obtain the circular measure of the angle.

Ex. Find the circular measure of $45^\circ.\,15'$;

$$45^\circ.\,15' = 45\cdot 25 \text{ degrees;}$$

$\therefore$ circular measure required is

$$\frac{45\cdot 25 \times \pi}{180} = \frac{4525\pi}{18000} = \frac{905\pi}{3600} = \frac{181\pi}{720}.$$

EXAMPLES.—VIII.

Express in circular measure the following angles:

(1) 60°. (2) $22^\circ.30'$. (3) $11^\circ.15'$. (4) 270°.
(5) 315°. (6) $24^\circ.13'$. (7) $95^\circ.20'$. (8) $12^\circ.5'.4''$.
(9) The angles of an equilateral triangle.
(10) The angles of an isosceles right-angled triangle.

42. If the circular measure of an angle be given to find the number of degrees which it contains.

Let θ be the given circular measure.

$$1^c = \frac{1}{\pi} \text{ of two right angles}$$

$$= \frac{1}{\pi} \text{ of } 180 \text{ degrees}$$

$$= \frac{180}{\pi} \text{ degrees};$$

$$\therefore\ \theta^c = \frac{\theta . 180}{\pi} \text{ degrees.}$$

Hence we obtain the following rule:

If an angle be expressed in circular measure, multiply the measure by 180, divide the result by π, and you obtain the measure of the angle in degrees.

Ex. Express in degrees the angle whose circular measure is $\frac{5\pi}{8}$.

The measure in degrees $= \frac{5\pi \times 180}{8 \times \pi} = \frac{900}{8} = 112\cdot5$ degrees.

EXAMPLES.—IX.

Express in degrees, &c. the angles whose circular measures are

(1) $\frac{\pi}{2}$. (2) $\frac{\pi}{3}$. (3) $\frac{\pi}{4}$. (4) $\frac{\pi}{6}$. (5) $\frac{2\pi}{3}$.
(6) $\frac{1}{2}$. (7) $\frac{1}{3}$. (8) $\frac{1}{4}$. (9) $\frac{1}{6}$. (10) $\frac{2}{3}$.

43. Similar rules will hold with respect to the equations for connecting the centesimal and the circular systems, 200 being put in the place of 180: thus

$$\text{circular measure of an angle containing } G \text{ grades} = G \cdot \frac{\pi}{200},$$

$$\text{number of grades in the angle whose circular measure is } \theta = \theta \cdot \frac{200}{\pi}.$$

EXAMPLES.—X.

Express in circular measure the following angles:

(1) 50^{g}. (2) 25^{g}. (3) $6^{g}.\,25^{\backprime}$. (4) 250^{g}.
(5) 500^{g}. (6) $13^{g}.\,5^{\backprime}.\,5^{\backprime\backprime}$. (7) $24^{g}.\,15^{\backprime}.\,2^{\backprime\backprime}\cdot 15$.
(8) $125^{g}.\,0^{\backprime}.\,13^{\backprime\backprime}$. (9) $3^{\backprime}$. (10) $5^{\backprime\backprime}$.

EXAMPLES.—XI.

Express in grades, &c. the angles whose circular measures are

(1) $\frac{\pi}{3}$. (2) $\frac{\pi}{5}$. (3) $\frac{\pi}{6}$. (4) $\frac{2\pi}{3}$. (5) $\frac{3\pi}{5}$.
(6) $\frac{1}{3}$. (7) $\frac{1}{5}$. (8) $\frac{1}{8}$. (9) $\frac{3}{5}$. (10) $2\frac{3}{10}$.

44. We shall now give a set of Miscellaneous Examples to illustrate the principles explained in this and the two preceding Chapters.

EXAMPLES.—XII.

1. If the unit of angular measurement be 5^{0}, what is the measure of $22\frac{1}{2}^{0}$?

2. If an angle of $42\frac{1}{2}^{0}$ be represented by 10, what is the unit of measurement?

3. An angle referred to different units has measures in the ratio 8 to 5; one unit is 2^{0}, what is the other? Express each unit in terms of the other.

4. An angle referred to different units has measures in the ratio 7 to 6; one unit is 3°, what is the other? Express each unit in terms of the other.

5. If half a right angle be taken as the unit of angular measurement, what is the measure of an angle of 42°?

6. Compare the angles $13^\circ.\,13'.\,48''$ and $14^g.\,7^{\backprime}$.

7. If D be the number of degrees in any angle and G the number of grades, shew that $G = D + \frac{1}{9} D$.

8. An equilateral triangle is divided into two triangles by a line bisecting one side; express the angles of these two triangles in degrees and grades respectively.

9. It being given that the angle subtended by an arc equal to the radius is $57^\circ{\cdot}29577$, find the ratio of the circumference of a circle to its diameter.

10. Two angles of a triangle are in magnitude as 2 : 3. If the third angle be a right angle, express the angles of this triangle in each of the three systems of measurement.

11. Two straight lines drawn from the centre of a circle contain an angle subtended by an arc which is to the whole circumference as 13 : 27; express this angle in degrees.

12. An arc of a circle is to the whole circumference as 17 : 54; express in grades the angle which the arc subtends at the centre of the circle.

13. If the angles of a triangle are in Arithmetical Progression, shew that one of them is 60°.

14. Determine in grades the magnitude of the angle subtended by an arc two feet long at the centre of a circle whose radius is 18 inches.

15. Compare the measures of $23^\circ.\,5'$ when expressed in centesimal and circular measure, taking $\pi = \frac{22}{7}$.

16. If a degree be represented by π, what is the unit of angular measurement?

17. What is the circular measure of $11^g.\ 30^\backprime$ if $\pi = \frac{355}{113}$?

18. Reduce $39^g{\cdot}012$ to degrees, minutes and seconds.

19. If the numerical value of an angle measured by the circular system be $\left(\frac{\pi}{3}\right)^2$, how many degrees does it contain?

20. The whole circumference of one circle is just long enough to subtend an angle of one grade at the centre of another circle: what part of the latter circumference will subtend an angle of one degree at the centre of the former circle?

21. Taking 4 right angles as the unit, what number will represent 1°, 1^g, 1^c respectively?

22. If there be m English minutes in an angle, find the number of French seconds in the same angle.

23. What is the number of grades in the unit of circular measure, if $\pi = \frac{22}{7}$?

24. What fraction of a right angle must be the unit in order that an angle of $5^\circ.\ 33'.\ 20''$ may be represented by 5?

25. If half a right angle be the unit of angular measurement, express the angles whose measures are

$$\frac{3}{2},\ 4,\ \pi,\ 4n + \frac{1}{3},$$

(1) in degrees, (2) in units of circular measure.

26. What must be the unit angle if the sum of the measures of a degree and a grade is 1?

27. If the unit be an angle subtended at the centre of a circle by an arc three times as large as the radius, what number will represent an angle of 45°?

36. The units in the three systems when expressed in terms of one common standard, two right angles, stand thus:

the unit in the Sexagesimal Method $= \frac{1}{180}$ of two right angles,

the unit in the Centesimal Method $= \frac{1}{200}$ of two right angles,

the unit in the Circular Method $= \frac{1}{\pi}$ of two right angles.

37. It is not usual to assign any distinguishing mark to angles estimated by the Third Method, but for the purpose of stating the relation between the three units in a clear and concise form, we shall use the symbol 1^c to express the unit of circular measure.

Then we express the relation between the units thus:

$$1^\circ : 1^g : 1^c = \frac{1}{180} : \frac{1}{200} : \frac{1}{\pi}.$$

CHAPTER III.

ON THE METHOD OF CONVERTING THE MEASURES OF ANGLES FROM ONE TO ANOTHER SYSTEM OF MEASUREMENT.

38. We proceed to explain the process for converting the measures of angles from each of the three systems of measurement described in Chap. II. to the other two.

39. To convert the measure of an angle expressed in degrees to the corresponding measure in grades.

Let the given angle contain D degrees.

$$1 \text{ degree} = \frac{1}{90} \text{ of a right angle};$$

$$\therefore D \text{ degrees} = \frac{D}{90} \text{ of a right angle}$$

$$= \frac{D}{90} \text{ of } 100 \text{ grades}$$

$$= \frac{100D}{90} \text{ grades}$$

$$= \frac{10D}{9} \text{ grades.}$$

Hence we obtain the following rule: If an angle be expressed in degrees, multiply the measure in degrees by 10, divide the result by 9, and you obtain the measure of the angle in grades.

Ex. How many grades are contained in the angle $24^\circ. 51'. 45''$?

$$24^\circ. 51'. 45'' = 24{\cdot}8625 \text{ degrees.}$$

$$\begin{array}{r|l} & 10 \\ \hline 9 & 248{\cdot}625 \\ \hline \text{grades} & 27{\cdot}625 \end{array}$$

$\therefore$ the angle contains $27^g. 62^{\backprime}. 50^{\backprime\backprime}$.

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$$(\sin A)^2 \text{ is written thus, } \sin^2 A,$$
$$(\cos A)^2 \text{ is written thus, } \cos^2 A,$$

and so for the other ratios.

57. The angle of reference may be one of the angles of the triangle of reference, as it is in fig. 1 of Art. 51, and it will be found that this is the case with which we are chiefly concerned.

It may be an exterior angle of the triangle of reference, as in fig. 2.

It may be greater than two right angles, partly within and partly out of the triangle of reference, as in fig. 3.

Finally, it may be an angle greater than two right angles and out of the triangle of reference, as in fig. 4.

In the next articles we shall describe the method of determining the Trigonometrical Ratios in any particular case by inspection. Strictly our remarks will be applicable to all the figures of Art. 51, but the student may at first regard them as having especial reference to fig. 1.

58. The six Trigonometrical Ratios are *arithmetical* quantities, denoting the relations existing between the sides of a right-angled triangle, which we call the Triangle of Reference, taken two by two.

59. Let us now look at the *order* in which the sides are taken to form the ratios which we call the *sine* and the *cosine*.

Referring to the diagrams of Art. 51,

$$\text{sine } EQP = \frac{PM}{QP},$$

$$\text{cosine } EQP = \frac{QM}{QP}.$$

In each case the *denominator* of the fraction is formed by that side of the triangle of reference which is *opposite to the right angle.*

Of the two other sides we may call PM the side *opposite* to the angle of reference so as to distinguish it from QM the side *adjacent* to the angle of reference.

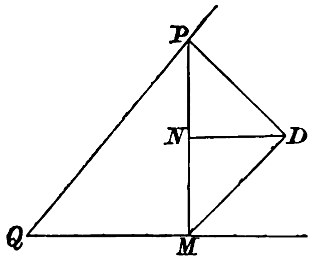
Hence in determining the ratio which we call the *sine* of a given angle we must take as the *numerator* of the fraction that side of our triangle of reference which is *opposite* to the given angle; and in determining the ratio which we call the *cosine* of a given angle we must take as the *numerator* of the fraction that side of our triangle of reference which is *adjacent* to the given angle: the denominator being in both cases that side of the triangle which *subtends the right angle* of the triangle.

60. If the remarks given in the preceding Article be clearly understood, the student will find no difficulty in writing down the ratios for the sine and the cosine of a given angle, whatever may be the position in which the triangle of reference for that angle may stand.

Suppose PM to be perpendicular to QM,

DN to be perpendicular to PM.

Then PQM, PDN, DMN are three right-angled triangles.



Now
$$\sin PQM = \frac{PM}{PQ} : \quad \cos PQM = \frac{QM}{QP}.$$

$$\sin QPM = \frac{QM}{QP} : \quad \cos QPM = \frac{PM}{QP}.$$

$$\text{Also}\qquad \sin DPN = \frac{DN}{PD} : \quad \cos DPN = \frac{PN}{PD}.$$

$$\sin PDN = \frac{PN}{PD} : \quad \cos PDN = \frac{DN}{PD}.$$

$$\text{And}\qquad \sin DMN = \frac{DN}{DM} : \quad \cos DMN = \frac{MN}{DM}.$$

$$\sin NDM = \frac{MN}{DM} : \quad \cos NDM = \frac{DN}{DM}.$$

61. When once the student has acquired facility in fixing on the lines which form the ratios called the *sine* and the *cosine*, he will be able to determine the other four ratios without any trouble.

Examples.—XIII.

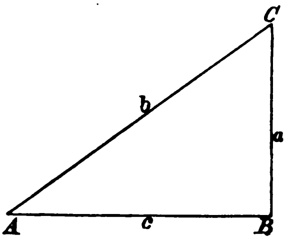
(1) Let ABC be a triangle. Draw from B a perpendicular BD on AC, and let it be *within* the triangle. Then write the following ratios:

$$\sin BAD, \quad \cos BAD, \quad \tan BAD;$$
$$\sin ABD, \quad \cot ABD, \quad \operatorname{cosec} ABD;$$
$$\sin BCD, \quad \sin CBD, \quad \tan BCD.$$

(2) Let OPQ be a triangle. Draw from O a perpendicular OR on PQ produced, so that it falls *without* the triangle. Then write the following ratios:

$$\sin OPQ, \quad \cos OPQ, \quad \cot OPQ;$$
$$\sin OQP, \quad \cos OQP, \quad \tan OQP;$$
$$\operatorname{cosec} OQR, \quad \cot QOR, \quad \sec OQR.$$

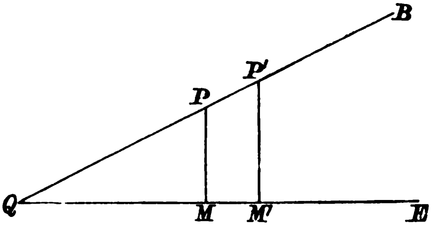
62. Let ABC be a right-angled triangle, having B as the right angle, and let the angles be denoted by the letters A, B, C, and the sides respectively opposite to them by the letters a, b, c.



The following results are worthy of notice, as being of frequent use.

$$a = b \,.\, \sin A \ = b \,.\, \cos C = c \,.\, \tan A = c \,.\, \cot C,$$
$$b = a \,.\, \operatorname{cosec} A = a \,.\, \sec C = c \,.\, \sec A = c \,.\, \operatorname{cosec} C,$$
$$c = a \,.\, \cot A \ = a \,.\, \tan C = b \,.\, \cos A = b \,.\, \sin C.$$

63. *The trigonometrical ratios remain unchanged so long as the angle is the same.*



Let EQB be any angle.

In QB take any points P, P' and draw PM, $P'M'$ at right angles to QE.

Then, since PM, $P'M'$ are parallel,

the triangles QPM, $QP'M'$ are similar.

Hence $$\frac{PM}{QP} = \frac{P'M'}{QP'};$$

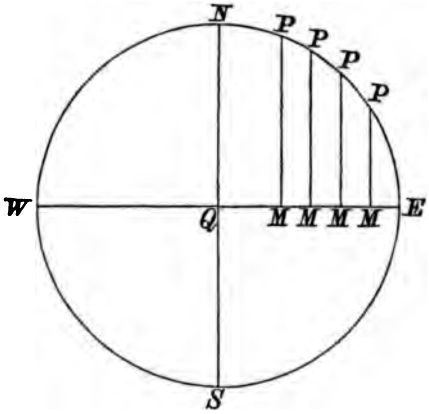
i. e. the value of the sine of EQB is the same so long as the angle is the same, and this result holds good for the other ratios.

The figure represents the simplest case, where the given angle is less than a right angle, but the conclusion is true for all angles.

CHAPTER V.

ON THE CHANGES IN SIGN AND MAGNITUDE OF THE TRIGONOMETRICAL RATIOS OF AN ANGLE AS IT INCREASES FROM 0° TO 360°.

64. Let NS, WE bisect each other at right angles in the point Q, and let a line equal in length to QE be supposed to revolve from QE round the fixed point Q.



As the revolving line passes from the position QE to the positions QN, QW, QS, QE, the extremity traces out a circle $ENWS$.

If we take a succession of points in EN, as P, P, P, P, and from them let fall perpendiculars PM, PM, PM, PM on the line QE, and do the same in the other quadrants, it is clear that

In passing from E to N,

PM continually increases from zero to QN,

QM continually decreases from QE to zero.

In passing from N to W,

PM continually decreases from QN to zero,

QM continually increases from zero to QW.

In passing from W to S,

PM continually increases from zero to QS,

QM continually decreases from QW to zero.

In passing from S to E,

PM continually decreases from QS to zero,

QM continually increases from zero to QE.

Again,

PM is positive in the first and second quadrants,

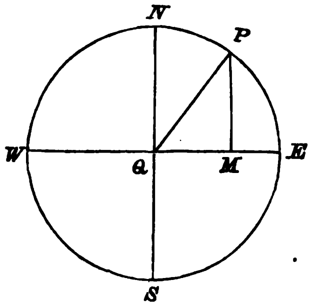
negative in the third and fourth.

QM is positive in the first and fourth quadrants,

negative in the second and third.

QP is always positive.

65. To trace the changes in sign and magnitude of the sine of an angle as the angle changes from 0° to 360°.



Let NQS, EQW be two diameters of a circle at right angles.

Let a radius QP by revolving from QE trace out any angle PQE, and denote this angle by A.

From P draw PM at right angles to EQW.

Then $$\sin A = \frac{PM}{QP}.$$

As A changes from 0° to 90°, QP revolves from QE to QN;

∴ PM changes from 0 to QN and is positive,
QP .... QE to QN .... positive;
∴ $\sin A$ .... 0 to 1 .... positive.

As A changes from 90° to 180°, QP revolves from QN to QW;

∴ PM changes from QN to 0 and is positive,
QP .... QN to QW .... positive;
∴ $\sin A$ .... 1 to 0 .... positive.

As A changes from 180° to 270°, QP revolves from QW to QS;

∴ PM changes from 0 to QS and is negative,
QP .... QW to QS .... positive;
∴ $\sin A$ .... 0 to 1 .... negative.

As A changes from 270° to 360°, QP revolves from QS to QE;

∴ PM changes from QS to 0 and is negative,
QP .... QS to QE .... positive;
∴ sin A .... 1 to 0 .... negative.

66. To trace the changes in the sign and magnitude of the cosine of an angle as the angle changes from 0° to 360°.

Making the same construction as in Art. 65,

$$\cos A = \frac{QM}{QP}.$$

As A changes from 0° to 90°, QP revolves from QE to QN;

∴ QM changes from QE to 0 and is positive,
QP .... QE to QN .... positive;
∴ cos A .... 1 to 0 .... positive.

As A changes from 90° to 180°, QP revolves from QN to QW;

∴ QM changes from 0 to QW and is negative,
QP .... QN to QW .... positive;
∴ cos A .... 0 to 1 .... negative.

As A changes from 180° to 270°, QP revolves from QW to QS;

∴ QM changes from QW to 0 and is negative,
QP .... QW to QS .... positive;
∴ cos A .... 1 to 0 .... negative.

As A changes from 270° to 360°, QP revolves from QS to QE;

∴ QM changes from 0 to QE and is positive,
QP .... QS to QE .... positive;
∴ cos A .... 0 to 1 .... positive.

67. To trace the changes in the sign and magnitude of the tangent of an angle as the angle changes from 0° to 360°.

Making the same construction as in Art. 65,

$$\tan A = \frac{PM}{QM}.$$

As A changes from 0° to 90°, QP revolves from QE to QN;

∴ PM changes from 0 to QN and is positive,
QM .... QE to 0 .... positive;
∴ $\tan A$ .... 0 to ∞ .... positive.

As A changes from 90° to 180°, QP revolves from QN to QW;

∴ PM changes from QN to 0 and is positive,
QM .... 0 to QW...... negative;
∴ $\tan A$ .... ∞ to 0 ...... negative.

As A changes from 180° to 270°, QP revolves from QW to QS;

∴ PM changes from 0 to QS and is negative,
QM .... QW to 0 ...... negative;
∴ $\tan A$ .... 0 to ∞ ...... positive.

As A changes from 270° to 360°, QP revolves from QS to QE;

∴ PM changes from QS to 0 and is negative,
QM .... 0 to QE ...... positive;
∴ $\tan A$ .... ∞ to 0 ...... negative.

68. The changes of the cosecant, secant and cotangent should be traced for himself by the student for practice.

69. We now present the changes of the Trigonometrical Ratios in a convenient tabular form.

Columns 1, 3, 5, 7, 9 give the values of the ratios for the particular values of the angle placed above the columns.

Columns 2, 4, 6, 8 give the signs of the ratios as the angle passes from 0 to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°.

A	0°		90°		180°		270°		360°
$\sin A$	0	+	1	+	0	−	−1	−	0
$\cos A$	1	+	0	−	−1	−	0	+	1
$\tan A$	0	+	∞	−	0	+	∞	−	0
$\operatorname{cosec} A$	∞	+	1	+	∞	−	−1	−	∞
$\sec A$	1	+	∞	−	−1	−	∞	+	1
$\cot A$	∞	+	0	−	∞	+	0	−	∞

70. From the results given in this table we are led to the following conclusion, which will be more fully explained hereafter.

If the value of a Trigonometrical Ratio be given, we cannot fix on one angle to which it exclusively belongs.

Thus if the given value of $\sin A$ be $\frac{1}{2}$, we know, since $\sin A$ passes through all values from 0 to 1 as A increases from 0° to 90°, that *one* value of A lies between 0° and 90°. But since we also know that the value of $\sin A$ passes through all values between 1 and 0 as A increases from 90° to 180°, it is evident that there is *another* value of A between 90° and 180° for which

$$\sin A = \frac{1}{2}.$$

CHAPTER VI.

ON RATIOS OF ANGLES IN THE FIRST QUADRANT.

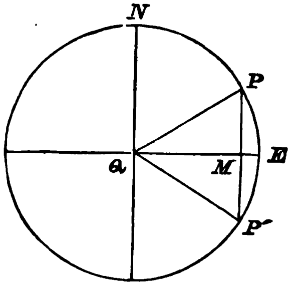
71. We have now to treat of the values of the Trigonometrical Ratios for some particular angles in the first quadrant. These angles, which we shall take in their proper order, as they are traced out by the revolving line, are 0°, 30°, 45°, 60°, 90°.

72. The signs of all the ratios for an angle in the first quadrant are positive.

73. To find the Trigonometrical Ratios for an angle of 0°. We have already proved in the preceding Chapter that

$$\sin 0^\circ = 0, \qquad \cos 0^\circ = 1, \qquad \tan 0^\circ = 0.$$
$$\operatorname{cosec} 0^\circ = \infty, \qquad \sec 0^\circ = 1, \qquad \cot 0^\circ = \infty.$$

74. To find the Trigonometrical Ratios for an angle of 30°.



Let QP revolving from the position QE describe an angle EQP equal to one-third of a right angle, that is an angle of 30°.

Draw the chord PMP' at right angles to QE, and join QP'.

Then angle $QP'P = QPP' = 90^\circ - PQM = 60^\circ$.

Thus PQP' is an equilateral triangle, and QM bisects PP';

$$\therefore\ QP = 2PM.$$

Let the measure of PM be m.

Then the measure of QP is $2m$.

And the measure of QM is $\sqrt{(4m^2 - m^2)} = \sqrt{(3m^2)} = m \, . \sqrt{3}$.

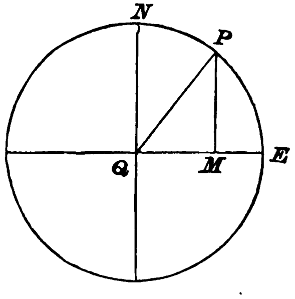
Then $$\sin 30^0 = \frac{PM}{PQ} = \frac{m}{2m} = \frac{1}{2},$$

$$\cos 30^0 = \frac{QM}{PQ} = \frac{m\sqrt{3}}{2m} = \frac{\sqrt{3}}{2},$$

$$\tan 30^0 = \frac{PM}{QM} = \frac{m}{m\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

So also $$\operatorname{cosec} 30^0 = 2,\quad \sec 30^0 = \frac{2}{\sqrt{3}},\quad \cot 30^0 = \sqrt{3}.$$

75. To find the Trigonometrical Ratios for an angle of 45°.



Let QP revolving from the position QE describe an angle EQP equal to half a right angle, that is an angle of 45°.

Draw PM at right angles to QE.

Then since PQM and QPM are together equal to a right angle, and PQM is half a right angle, QPM is also half a right angle.

Thus PQM is an isosceles triangle, and $QM = MP$.

Let the measure of QM be m.

Then the measure of PM is m.

And the measure of QP is $\sqrt{(m^2+m^2)} = \sqrt{(2m^2)} = m\sqrt{2}$.

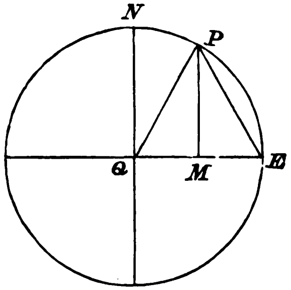
Then $$\sin 45^\circ = \frac{PM}{QP} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{QM}{QP} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\tan 45^\circ = \frac{PM}{QM} = \frac{m}{m} = 1.$$

So also $\operatorname{cosec} 45^\circ = \sqrt{2}$, $\sec 45^\circ = \sqrt{2}$, $\cot 45^\circ = 1$.

76. To find the Trigonometrical Ratios for an angle of 60°.



Let QP revolving from the position QE describe an angle EQP equal to two-thirds of a right angle, that is an angle of 60°.

Draw PM at right angles to QE and join PE.

Then PQE is an equilateral triangle and PM bisects QE;

$$\therefore\ QP = 2QM.$$

Let the measure of QM be m.

Then the measure of QP is $2m$.

And the measure of PM is $\sqrt{(4m^2 - m^2)} = \sqrt{(3m^2)} = m\sqrt{3}$.

Then $$\sin 60^\circ = \frac{PM}{QP} = \frac{m\sqrt{3}}{2m} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{QM}{QP} = \frac{m}{2m} = \frac{1}{2},$$

$$\tan 60^\circ = \frac{PM}{QM} = \frac{m\sqrt{3}}{m} = \sqrt{3}.$$

So also $\quad \operatorname{cosec} 60^\circ = \dfrac{2}{\sqrt{3}}, \quad \sec 60^\circ = 2, \quad \cot 60^\circ = \dfrac{1}{\sqrt{3}}.$

77. To find the Trigonometrical Ratios for an angle of 90°.

We have already proved in the preceding Chapter that

$$\sin 90^\circ = 1, \quad \cos 90^\circ = 0, \quad \tan 90^\circ = \infty,$$

$$\operatorname{cosec} 90^\circ = 1, \quad \sec 90^\circ = \infty, \quad \cot 90^\circ = 0.$$

Examples.—XIV.

If $\alpha = 0^\circ$, $\beta = 30^\circ$, $\gamma = 45^\circ$, $\delta = 60^\circ$, $\theta = 90^\circ$, find the numerical values of the following expressions:

(1) $\cos\alpha \,.\, \sin\gamma \,.\, \cos\delta.$ (2) $\sin\theta \,.\, \cos\frac{\pi}{4} \,.\, \operatorname{cosec}\delta.$

(3) $\sin\frac{\pi}{2} + \cos\frac{\pi}{6} - \sec\alpha.$ (4) $\sin\frac{\pi}{3} \,.\, \operatorname{cosec}\frac{\pi}{2} \,.\, \sec\delta.$

(5) $(\sin\theta - \cos\theta + \operatorname{cosec}\beta)\left(\cos\theta + \sec\frac{\pi}{4} + \cot\delta\right).$

Also prove the following:

(6) $(\sin\delta - \sin\gamma)(\cos\beta + \cos\gamma) = \sin^2\beta.$

(7) $$\cot^2\frac{\pi}{4} - \cot^2\frac{\pi}{6} = \frac{\sin^2\frac{\pi}{6} - \sin^2\frac{\pi}{4}}{\sin^2\frac{\pi}{4} \,.\, \sin^2\frac{\pi}{6}}.$$

(8) $\left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3} - \cos\frac{\pi}{3}\right) = \cos\frac{\pi}{3}.$

(9) $\cos\frac{\pi}{3} \,.\, \cos\frac{\pi}{6} = \frac{1}{2}\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right).$

(10) $\tan^2\frac{\pi}{3} - \tan^2\frac{\pi}{6} = \dfrac{\sin^2\frac{\pi}{3} - \sin^2\frac{\pi}{6}}{\cos^2\frac{\pi}{3}\,.\,\cos^2\frac{\pi}{6}}$.

78. We are now able to give some simple examples of the practical use of Trigonometry in the measurement of heights and distances.

79. The values of the sines, cosines, tangents and the other ratios have been calculated for all angles succeeding each other at intervals of 1′, and the results registered in Tables.

Instruments have been invented for determining:

1. The angle which the line joining two distant objects subtends at the eye of the observer.

2. The angle which a line joining the eye of the observer and a distant object makes with the horizontal plane.

If the object be *above* the observer the angle is called the Angle of *Elevation.*

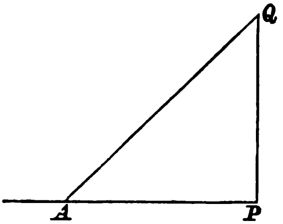
If the object be *below* the observer the angle is called the Angle of *Depression.*

80. To find the height of an object standing on a horizontal plane, the base of the object being accessible.

Let PQ be a vertical column.

From the base P measure a horizontal line AP.

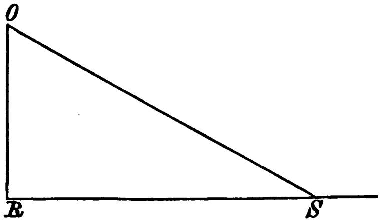
Then observe the angle of elevation QAP.



We can then determine the height of the column, for

$$QP = AP\,.\,\tan QAP.$$

81. To find the breadth of a river.



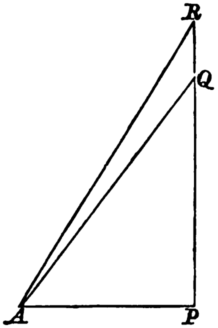
Let RS be the horizontal line joining two objects on the opposite banks.

From O, a point in a vertical line with R, observe the angle of depression OSR.

Then if OR be measured we can determine the length of RS, for

$$RS = \frac{OR}{\tan OSR}.$$

82. To find the height of a flag-staff on the top of a tower.



Let RQ be the flag-staff.

From P the base of the tower measure a horizontal line AP.

Observe the angles RAP and QAP.

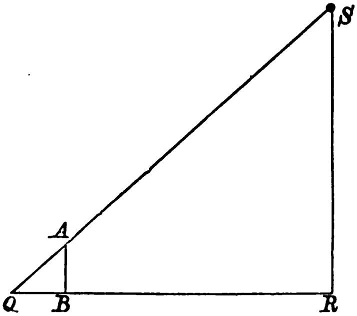
Then we can find the length of RQ, for

$$\begin{aligned} RQ &= RP - QP \\ &= AP . \tan RAP - AP . \tan QAP. \end{aligned}$$

83. To find the altitude of the Sun.

The altitude of the Sun is measured by the angle between a horizontal line and a line passing through the centre of the Sun.

If AB be a stick standing at right angles to the horizontal plane QR, and QB the shadow of the stick on the horizontal plane, a line joining QA will pass through the centre of the Sun.



Then if we measure AB and QB we shall know the altitude of the Sun, for

$$\tan SQR = \frac{AB}{QB}.$$

Examples.—XV.

(1) At a point 200 feet from a tower and on a level with its base the angle of elevation of its summit is found to be 60°: what is the height of the tower?

(2) What is the height of a tower whose top appears at an elevation of 30° to an observer 140 feet from the foot of the tower on a horizontal plane, his eye being 5 feet from the ground?

(3) Determine the altitude of the Sun when the length of a vertical stick is to the length of the shadow of the stick as $\sqrt{3} : 1$.

(4) At 300 feet measured horizontally from the foot of a steeple the angle of elevation of the top is found to be 30°. What is the height of the steeple?

(5) From the top of a rock 245 feet above the sea the angle of depression of a ship's hull is found to be 30°. How far is the ship distant?

(6) From the top of a hill there are observed two consecutive milestones, on a horizontal road, running from the base. The angles of depression are found to be 45° and 30°. Find the height of the hill.

(7) A flag-staff stands on a tower. I measure from the bottom of the tower a distance of 100 feet. I then find that the top of the flag-staff subtends an angle of 45° and the top of the tower an angle of 30° at my place of observation. What is the height of the flag-staff?

(8) From the summit of a tower whose height is 108 feet, the angles of depression of the top and bottom of a vertical column, standing on a level with the base of the tower, are found to be 30° and 60° respectively. Find the height of the column.

(9) A person observes the elevation of a tower to be 60°, and on receding from it 100 yards further he finds the elevation to be 30°. Required the height of the tower.

(10) A stick 10 feet in length is placed vertically in the ground, and the length of its shadow is 25 feet. Find the altitude of the Sun, having given tan 25° = ·4.

(11) A spire stands on a tower in the form of a cube whose edge is 35 feet. From a point 23 feet above the level of the base of the tower, and 20 yards distant from the tower, the elevation of the top of the spire is found to be 56°. 34′. Find the height of the top of the spire, having given

$$\tan 56^\circ.\,34' = 1{\cdot}5.$$

(12) The length of a kite string is 250 yards and the angle of elevation of the kite is 30°. Find the height of the kite.

(13) The height of a house-top is 60 feet. A rope is stretched from it and is inclined at an angle of 40°. 30′ to the ground. Find the length of the rope, having given sin 40°. 30′ = ·65.

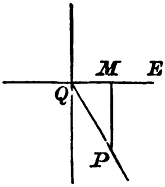
(14) A tower on the bank of a river is 120 feet high, and the angle of elevation of the top of the tower from the opposite bank is 20°; find the breadth of the river, having given tan 20° = ·35.

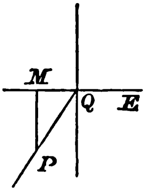
(15) The altitude of the Sun is 36°. 30′: what is the length of the shadow of a man 6 feet high, if tan 36°. 30′ = ·745?

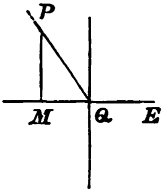
CHAPTER VII.

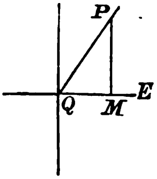
ON THE RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS FOR THE SAME ANGLE.

84. Let EQP be any angle traced by QP revolving from the position QE, and let a perpendicular PM be dropped on QE or EQ produced, thus









Let the angle EQP be denoted by A.

Then we can prove the following relations:

I. $\tan A = \dfrac{\sin A}{\cos A}$.

$$\text{For } \tan A = \frac{PM}{QM} = \frac{\dfrac{PM}{QP}}{\dfrac{QM}{QP}} = \frac{\sin A}{\cos A}.$$

II. $\sin^2 A + \cos^2 A = 1$.

$$\text{For } \sin^2 A + \cos^2 A = \frac{PM^2}{QP^2} + \frac{QM^2}{QP^2} = \frac{PM^2 + QM^2}{QP^2} = \frac{QP^2}{QP^2} = 1.$$

III. $\sec^2 A = 1 + \tan^2 A$.

For $\sec^2 A = \frac{QP^2}{QM^2} = \frac{QM^2 + PM^2}{QM^2} = 1 + \frac{PM^2}{QM^2} = 1 + \tan^2 A$.

IV. $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

For $\operatorname{cosec}^2 A = \frac{QP^2}{PM^2} = \frac{PM^2 + QM^2}{PM^2} = 1 + \frac{QM^2}{PM^2} = 1 + \cot^2 A$.

85. We shall now give a number of easy examples by which the student may become familiar with the formulæ which we have just obtained.

He must observe that these formulæ hold good for all magnitudes of the angle which we have represented by the letter A, that is, not only

$$\sin^2 A + \cos^2 A = 1,$$

but also $$\sin^2 \theta + \cos^2 \theta = 1,$$

and $$\sin^2 x + \cos^2 x = 1,$$

and $$\sin^2 45^\circ + \cos^2 45^\circ = 1,$$

and $$\sin^2 60^g + \cos^2 60^g = 1,$$

and $$\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2} = 1.$$

And similarly for the other formulæ.

86. If then any angle be represented by θ, we know from Art. 84,

$$(1) \quad \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

$$(2) \quad \sin^2 \theta + \cos^2 \theta = 1.$$

$$(3) \quad \sec^2 \theta = 1 + \tan^2 \theta.$$

$$(4) \quad \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta.$$

And we also know from Art. 52,

$$(5) \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}.$$

$$(6) \quad \sec \theta = \frac{1}{\cos \theta}.$$

$$(7) \quad \cot \theta = \frac{1}{\tan \theta}.$$

87. Ex. 1. Shew that $\sec\theta - \tan\theta \,.\, \sin\theta = \cos\theta$.

$$\sec\theta - \tan\theta \,.\, \sin\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} \,.\, \sin\theta, \text{ by form (6) and (1),}$$

$$= \frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{1 - \sin^2\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta}{\cos\theta}, \text{ by (2),}$$

$$= \cos\theta.$$

Ex. 2. Shew that $\cot a - \sec a \operatorname{cosec} a (1 - 2\sin^2 a) = \tan a$.

$$\cot a - \sec a \,.\, \operatorname{cosec} a (1 - 2\sin^2 a)$$

$$= \frac{\cos a}{\sin a} - \frac{1}{\cos a} \,.\, \frac{1}{\sin a}(1 - 2\sin^2 a), \text{ by (7, 6, 5),}$$

$$= \frac{\cos a}{\sin a} - \frac{1}{\cos a \,.\, \sin a} + \frac{2\sin^2 a}{\cos a \,.\, \sin a}$$

$$= \frac{\cos^2 a - 1 + 2\sin^2 a}{\cos a \,.\, \sin a}$$

$$= \frac{\cos^2 a + \sin^2 a + \sin^2 a - 1}{\cos a \,.\, \sin a}$$

$$= \frac{1 + \sin^2 a - 1}{\cos a \,.\, \sin a}, \text{ by (1),}$$

$$= \frac{\sin^2 a}{\cos a \,.\, \sin a}$$

$$= \frac{\sin a}{\cos a}$$

$$= \tan a.$$

It will be observed that in working these examples we commenced by expressing the other ratios in terms of the sine and cosine, and the beginner will find this the simplest course in most cases.

EXAMPLES.—XVI.

Prove the following relations:

(1) $\cos\theta\,.\,\tan\theta=\sin\theta.$ (2) $\sin\theta\,.\,\cot\theta=\cos\theta.$

(3) $\sin a\,.\,\sec a=\tan a.$ (4) $\cos a\,.\,\operatorname{cosec} a=\cot a.$

(5) $(1+\tan^2\theta)\,.\,\cos^2\theta=1.$ (6) $(1+\cot^2\theta)\,.\,\sin^2\theta=1.$

(7) $\dfrac{\tan^2 a}{1+\tan^2 a}=\sin^2 a.$ (8) $\dfrac{\operatorname{cosec}^2 a-1}{\operatorname{cosec}^2 a}=\cos^2 a.$

(9) $\tan x+\cot x=\sec x\,.\,\operatorname{cosec} x.$

(10) $\dfrac{\cos x\,.\,\operatorname{cosec} x\,.\,\tan x}{\sin x\,.\,\sec x\,.\,\cot x}=1.$

(11) $\cos x+\sin x\,.\,\tan x=\sec x.$

(12) $\dfrac{\cos\theta}{\tan\theta\,.\,\cot^2\theta}=\sin\theta.$ (13) $(\cos^2\theta-1)(\cot^2\theta+1)=-1.$

(14) $\cot^2 a-\cos^2 a=\cot^2 a\,.\,\cos^2 a.$

(15) $\sec^2 a\,.\,\operatorname{cosec}^2 a=\sec^2 a+\operatorname{cosec}^2 a.$

(16) $\sin^2\phi+\sin^2\phi\,.\,\tan^2\phi=\tan^2\phi.$

(17) $\cot^2\phi\,.\,\sin^2\phi+\sin^2\phi=1.$

(18) $\sec^2\phi-1=\sin^2\phi\,.\,\sec^2\phi.$

(19) $\sin^2\phi=2\operatorname{versin}\phi-\operatorname{versin}^2\phi.$

(20) $\dfrac{\sec\theta-1}{\sec\theta}=\operatorname{versin}\theta.$

88. We shall next shew how to express the cosine, tangent and other ratios in terms of the sine.

Since $\sin^2\theta+\cos^2\theta=1,$

$$\cos^2\theta=1-\sin^2\theta;$$

$$\therefore\ \cos\theta=\pm\sqrt{(1-\sin^2\theta)}.$$

Again $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\pm\sqrt{(1-\sin^2\theta)}}.$$

Also $$\operatorname{cosec}\theta = \frac{1}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{(1-\sin^2\theta)}},$$

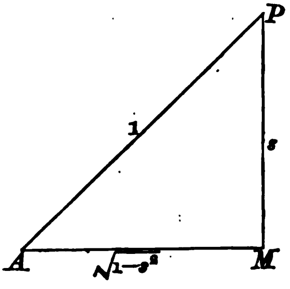
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\pm\sqrt{(1-\sin^2\theta)}}{\sin\theta}.$$

The double sign before the root-symbols is to be explained thus. For an assigned value of $\sin\theta$ we shall have more than one value of θ (Art. 70). Hence we have an ambiguity when we endeavour to find $\cos\theta$ from the known value of $\sin\theta$. The double sign may *generally* be omitted in the examples which we shall hereafter give.

89. We shall now give two examples of another method of arriving at expressions for the other ratios in terms of a particular ratio. These examples should be carefully studied.

(1) *To express the other trigonometrical ratios in terms of the sine.*

Let PAM be an angle whose sine is s, a numerical quantity.



Let PM be drawn perpendicular to AM.

Then if AP be represented by 1,

PM will be represented by s,

and AM will therefore be represented by $\sqrt{1-s^2}$.

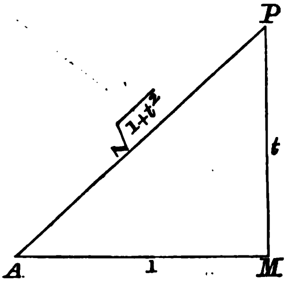
Then denoting PAM by A,

$$\cos A = \frac{AM}{AP} = \frac{\sqrt{1-s^2}}{1} = \sqrt{1-s^2} = \sqrt{1-\sin^2 A},$$

$$\tan A = \frac{PM}{AM} = \frac{s}{\sqrt{1-s^2}} = \frac{\sin A}{\sqrt{1-\sin^2 A}},$$

and similarly the other ratios may be expressed in terms of sin A.

(2) *To express the other trigonometrical ratios in terms of the tangent.*



Making the same construction as in the preceding Article,

Let $\tan A = t$.

Then if AM be represented by 1, PM will be represented by t; and AP will be represented by $\sqrt{1+t^2}$.

$$\text{Therefore } \sin A = \frac{PM}{AP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan A}{\sqrt{1+\tan^2 A}},$$

$$\cos A = \frac{AM}{AP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2 A}},$$

and similarly the other ratios may be found.

EXAMPLES.—XVII.

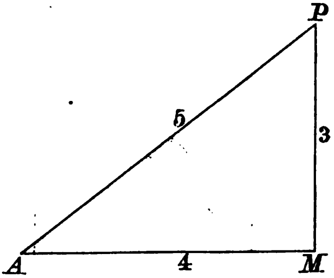
(1) Express the other trigonometrical ratios in terms of the cosine.

(2) Express the other trigonometrical ratios in terms of the cosecant.

(3) Express the other trigonometrical ratios in terms of the secant.

(4) Express the other trigonometrical ratios in terms of the cotangent.

90. *If any one of the Trigonometrical ratios be given the others may be found.*

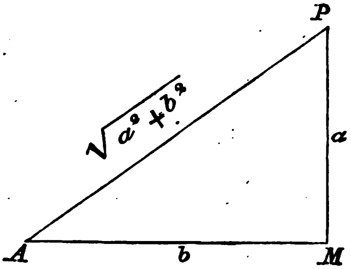


Thus suppose $\sin A = \frac{3}{5}$.

If PAM represent the angle, and PM be perpendicular to AM, we may represent PM by 3, AP by 5, and consequently AM by $\sqrt{25-9}$ or 4.

Then
$$\cos A = \frac{4}{5},$$
$$\tan A = \frac{3}{4},$$
$$\operatorname{cosec} A = \frac{5}{3},$$
$$\sec A = \frac{5}{4}.$$

91. If $\tan A = \frac{a}{b}$, to find $\sin A$ and $\cos A$.



If PAM represent the angle, and PM be perpendicular to AM, we may represent PM by a, AM by b, and consequently AP by $\sqrt{a^2+b^2}$.

Then
$$\sin A = \frac{a}{\sqrt{a^2+b^2}},$$
$$\cos A = \frac{b}{\sqrt{a^2+b^2}}.$$

EXAMPLES.—XVIII.

(1) Given $\sin \alpha = \frac{2}{3}$: find $\cos \alpha$ and $\tan \alpha$.

(2) Given $\cos \alpha = \frac{4}{5}$: find $\sin \alpha$ and $\tan \alpha$.

(3) Given $\operatorname{cosec} \theta = \frac{4}{3}$: find $\cos \theta$ and $\tan \theta$.

(4) Given $\sin \theta = \frac{1}{\sqrt{3}}$: find $\cos \theta$ and $\tan \theta$.

(5) Given $\tan \alpha = \frac{a^2}{b^2}$: find $\operatorname{cosec} \alpha$ and $\sec \alpha$.

(6) Given $\cos \alpha = \frac{a}{b}$: find $\tan \alpha$ and $\operatorname{cosec} \alpha$.

(7) Given $\sin\theta = a$: find $\tan\theta$ and $\sec\theta$.

(8) Given $\cos\theta = b$: find $\tan\theta$ and $\operatorname{cosec}\theta$.

(9) Given $\sin\theta = \cdot 6$: find $\cos\theta$ and $\cot\theta$.

(10) Given $\cos\theta = \cdot\dot{5}$: find $\cot\theta$ and $\operatorname{cosec}\theta$.

(11) Given $\operatorname{cosec}\theta = 2\cdot\dot{4}$: find $\cos\theta$ and $\cot\theta$.

(12) Given $\sec\theta = 1\cdot 0\dot{3}$: find $\sin\theta$ and $\tan\theta$.

(13) Given $\sin\phi = \frac{99}{101}$: find $\cos\phi$ and $\cot\phi$.

(14) Given $\cos\phi = \frac{20}{101}$: find $\sin\phi$ and $\tan\phi$.

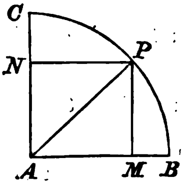
(15) Given $\operatorname{versin}\theta = \frac{1}{13}$: find $\sin\theta$ and $\sec\theta$.

92. We may here give the geometrical solutions of the problem of constructing an angle when its sine, cosine or tangent is given.

(1) Given that the sine of an angle is $\frac{a}{b}$, to construct the angle.

The sine of an angle cannot be greater than *unity*,

$\therefore$ a is not greater than b.



Draw a line $AB = b$, and describe a circle with centre A and radius AB.

Let BAC be a quadrant of this circle.

Mark off on AC the line $AN = a$.

Draw NP, PM at right angles to AC, AB.

Then PAM is the angle required: for

$$\sin PAM = \frac{PM}{AP} = \frac{AN}{AP} = \frac{a}{b}.$$

(2) Given that the cosine of an angle is $\frac{a}{b}$, to construct the angle.

Making the same construction as before,

PAN is the angle required: for

$$\cos PAN = \frac{AN}{AP} = \frac{a}{b}.$$

(3) Given that the tangent of an angle is $\frac{m}{n}$, to construct the angle.

Take a line $AM = n$, and draw $PM = m$ at right angles to AM.

Join AP. Then PAM is the angle required: for

$$\tan PAM = \frac{PM}{AM} = \frac{m}{n}.$$

93. We shall now give a set of examples similar to those in Ex. XVI. but presenting in some cases more difficulty.

EXAMPLES.—XIX.

Prove the following relations:

(1) $\sin A = \dfrac{1}{\sqrt{(1 + \cot^2 A)}}$. (2) $\cos A = \dfrac{1}{\sqrt{(1 + \tan^2 A)}}$.

(3) $\cos x = \dfrac{\cot x}{\sqrt{(1 + \cot^2 x)}}$. (4) $\tan x . \cos x = \sqrt{(1 - \cos^2 x)}$.

(5) $\cos \phi = \dfrac{\sqrt{(\operatorname{cosec}^2 \phi - 1)}}{\operatorname{cosec} \phi}$. (6) $\tan \phi = \sqrt{\left(\dfrac{1 - \cos^2 \phi}{\cos^2 \phi}\right)}$.

(7) $\sin^2 a = (1 + \cos a) . \operatorname{versin} a$.

(8) $\tan^2\alpha - \tan^2\beta = \dfrac{\cos^2\beta - \cos^2\alpha}{\cos^2\beta \,.\, \cos^2\alpha}$.

(9) $\cot^2\alpha - \cot^2\beta = \dfrac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha \,.\, \sin^2\beta}$.

(10) $\sin^2\theta \,.\, \tan^2\theta + \cos^2\theta \,.\, \cot^2\theta = \tan^2\theta + \cot^2\theta - 1$.

(11) $\sec^4\theta + \tan^4\theta = 1 + 2\sec^2\theta \,.\, \tan^2\theta$.

(12) $\operatorname{cosec}\theta\,(\sec\theta - 1) - \cot\theta\,(1 - \cos\theta) = \tan\theta - \sin\theta$.

(13) $\cot^2 b + \tan^2 b = \sin^2 b \operatorname{cosec}^2 b - 2$.

(14) $\cot^2 A - \cos^2 A = \cos^4 A \operatorname{cosec}^2 A$.

(15) $\tan^2\theta - \sin^2\theta = \sin^4\theta \sec^2\theta$.

(16) $(\sec\theta - \operatorname{cosec}\theta)\,(1 + \cot\theta + \tan\theta) = \dfrac{\sec^2\theta}{\operatorname{cosec}\theta} - \dfrac{\operatorname{cosec}^2\theta}{\sec\theta}$.

(17) $\dfrac{\operatorname{cosec}\theta}{\sec\theta} + \dfrac{\sec\theta}{\operatorname{cosec}\theta} = \sec\theta \,.\, \operatorname{cosec}\theta$.

(18) $\cos\theta\,(\tan\theta + 2)\,(2\tan\theta + 1) = 2\sec\theta + 5\sin\theta$.

(19) $\cos x\,(2\sec x + \tan x)\,(\sec x - 2\tan x) = 2\cos x - 3\tan x$.

(20) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$.

(21) $\dfrac{\sec\theta \,.\, \cot\theta - \operatorname{cosec}\theta \,.\, \tan\theta}{\cos\theta - \sin\theta} = \operatorname{cosec}\theta \,.\, \sec\theta$.

(22) $\sec\theta + \operatorname{cosec}\theta \,.\, \tan^3\theta\,(1 + \operatorname{cosec}^2\theta) = 2\sec^3\theta$.

(23) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = (1 + \sec\theta \,.\, \operatorname{cosec}\theta)^2$.

(24) $\dfrac{1 + (\operatorname{cosec}\theta \,.\, \tan\phi)^2}{1 + (\operatorname{cosec}\alpha \,.\, \tan\phi)^2} = \dfrac{1 + (\cot\theta \,.\, \sin\phi)^2}{1 + (\cot\alpha \,.\, \sin\phi)^2}$.

(25) $(3 - 4\sin^2 A)\,(1 - 3\tan^2 A) = (3 - \tan^2 A)\,(4\cos^2 A - 3)$.

CHAPTER VIII.

COMPARISON OF TRIGONOMETRICAL RATIOS FOR DIFFERENT ANGLES.

94. THE Complement of an angle is that angle which must be added to it to make a right angle.

Thus the complement of 60° is 30°, because $60^\circ + 30^\circ = 90^\circ$, and the complement of $14^\circ.\ 36'.\ 15''$ is $75^\circ.\ 23'.\ 45''$.

Also the complement of 80^g is 20^g, because $80^g + 20^g = 100^g$, and the complement of $42^g.\ 5^`.\ 28^{``}$ is $57^g.\ 94^`.\ 72^{``}$.

And the complement of $\frac{\pi}{6}$ is $\frac{\pi}{3}$, because $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$.

So generally, if α, β, γ be the measures of an angle in the three systems,

$$\text{complement of the angle} = 90^\circ - \alpha = 100^g - \beta = \frac{\pi}{2} - \gamma.$$

Hence if the angle be negative (see Art. 49), and its measures be $-\alpha$, $-\beta$, $-\gamma$ in the three systems,

$$\text{complement of the angle} = 90^\circ - (-\alpha) = 100^g - (-\beta) = \frac{\pi}{2} - (-\gamma)$$

$$= 90^\circ + \alpha = 100^g + \beta = \frac{\pi}{2} + \gamma.$$

EXAMPLES.—XX.

1. Find the complements of the following angles:

(1) $24^\circ.\ 14'.\ 42''$. (2) $43^\circ.\ 2'.\ 57''$.

(3) $64^\circ.\ 0'.\ 14''$. (4) $82^\circ.\ 4'.\ 15''$.

(5) $125^\circ.\ 15'.\ 42''$. (6) $178^\circ.\ 27'.\ 34''$.

(7) 195°. (8) 254°.

(9) -25°. (10) -245°.

2. Find the complements of the following angles:

(1) $32^{g}.\ 23^{`}.\ 24^{``}$. (2) $95^{g}.\ 3^{`}.\ 75^{``}$.

(3) $46^{g}.\ 0^{`}.\ 84^{``}$. (4) $2^{g}.\ 5^{`}.\ 4^{``}$.

(5) $135^{g}.\ 2^{`}.\ 5^{``}$. (6) $169^{g}.\ 0^{`}.\ 3^{``}$.

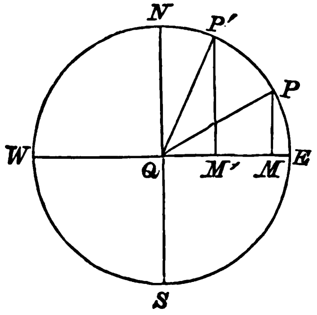
(7) 243^{g}. (8) 357^{g}.

(9) -35^{g}. (10) -245^{g}.

3. What are the complements of the following angles?

(1) $\frac{\pi}{4}$. (2) $\frac{\pi}{3}$. (3) $\frac{3\pi}{5}$. (4) $-\frac{\pi}{4}$. (5) $-\frac{3\pi}{4}$.

95. To compare the Trigonometrical Ratios of an angle and its complement.



Let NQS, EQW be two diameters of a circle at right angles.

Let a radius QP revolving from QE trace out the angle $PQE = A$.

Next, let the radius revolve from QE to QN and back again through an angle $NQP' = A$.

Then angle $EQP' = 90^\circ - A$.

Draw PM and $P'M'$ at right angles to EQ.

Now angle $QP'M' = NQP' = A = PQE$.

Hence the triangles $P'QM'$ and QPM are equal in all respects (Eucl. I. 26).

Therefore

$$\sin(90^\circ - A) = \sin EQP' = \frac{P'M'}{QP'} = \frac{QM}{QP} = \cos EQP = \cos A,$$

$$\cos(90^\circ - A) = \cos EQP' = \frac{QM'}{QP'} = \frac{PM}{QP} = \sin EQP = \sin A,$$

$$\tan(90^\circ - A) = \tan EQP' = \frac{P'M'}{QM'} = \frac{QM}{PM} = \cot EQP = \cot A.$$

And similarly it may be shewn that

$$\operatorname{cosec}(90^\circ - A) = \sec A,\quad \sec(90^\circ - A) = \operatorname{cosec} A,\quad \cot(90^\circ - A) = \tan A.$$

This is a proposition of great practical importance. We have only proved it for the case in which A is less than 90°, but the conclusions hold good for all values of A.

96. The Supplement of an angle is that angle which must be added to it to make two right angles.

Thus the supplement of 60° is 120°, because $60^\circ + 120^\circ = 180^\circ$, and the supplement of $24^\circ.\ 43'.\ 17''$ is $155^\circ.\ 16'.\ 43''$.

Also the supplement of 80^g is 120^g, because $80^g + 120^g = 200^g$, and the supplement of $114^g.\ 3^{\backprime}.\ 15^{\backprime\backprime}$ is $85^g.\ 96^{\backprime}.\ 85^{\backprime\backprime}$.

And the supplement of $\frac{\pi}{8}$ is $\frac{7\pi}{8}$, because $\frac{\pi}{8} + \frac{7\pi}{8} = \pi$.

So, generally, if α, β, γ be the measures of an angle in the three systems,

$$\text{supplement of the angle} = 180^\circ - \alpha = 200^g - \beta = \pi - \gamma.$$

Hence if the angle be negative and its measures be $-\alpha$, $-\beta$, $-\gamma$ in the three systems,

$$\begin{aligned}\text{supplement of the angle} &= 180^\circ - (-\alpha) = 200^g - (-\beta) = \pi - (-\gamma)\\ &= 180^\circ + \alpha = 200^g + \beta = \pi + \gamma.\end{aligned}$$

EXAMPLES.—XXI.

1. Find the supplements of the following angles:

(1)	$34^\circ.\,12'.\,49''$.	(2)	$132^\circ.\,24'.\,47''$.
(3)	$146^\circ.\,0'.\,41''$.	(4)	$28^\circ.\,15'.\,4''$.
(5)	$179^\circ.\,59'.\,59''$.	(6)	$100^\circ.\,49'.\,53''$.
(7)	245°.	(8)	$437^\circ.\,3'.\,4''$.
(9)	-49°.	(10)	-355°.

2. Find the supplements of the following angles:

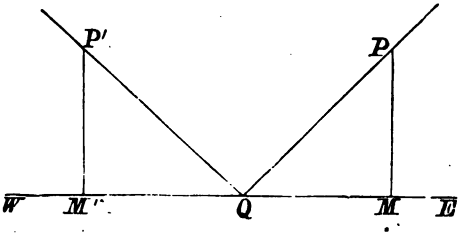
(1)	$132^g.\,32^{\backprime}.\,42^{\backprime\backprime}$.	(2)	$195^g.\,2^{\backprime}.\,57^{\backprime\backprime}$.
(3)	$3^g.\,97^{\backprime}.\,98^{\backprime\backprime}$.	(4)	$65^g.\,12^{\backprime}.\,8^{\backprime\backprime}$.
(5)	$154^g.\,3^{\backprime}.\,6^{\backprime\backprime}$.	(6)	$174^g.\,0^{\backprime}.\,4^{\backprime\backprime}$.
(7)	275^g.	(8)	$527^g.\,2^{\backprime}.\,14^{\backprime\backprime}$.
(9)	-35_g.	(10)	-325^g.

3. What are the supplements of the following angles?

(1) $\frac{\pi}{2}$. (2) $\frac{\pi}{3}$. (3) $\frac{4\pi}{5}$. (4) $-\frac{\pi}{4}$. (5) $-\frac{3\pi}{4}$.

4. Find the difference between the supplement of the complement of an angle and the complement of its supplement.

97. To compare the Trigonometrical Ratios of an angle and its supplement.



Let the angle $EQP = A$.

Produce EQ to W, and make the angle $P'QW = A$.

Take $QP' = QP$, and draw PM, $P'M'$ at right angles to EW.

Then the triangles PMQ, $P'M'Q$ are geometrically equal. (Eucl. I. 26.)

Therefore

$$\sin(180^\circ - A) = \sin EQP' = \frac{P'M'}{QP'} = \frac{PM}{QP} = \sin A,$$

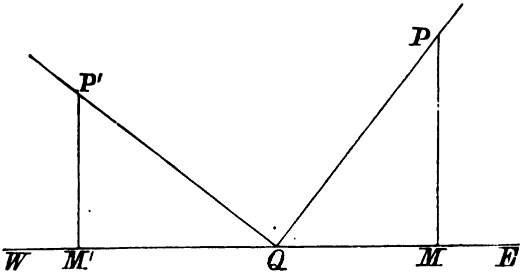
$$\cos(180^\circ - A) = \cos EQP' = \frac{QM'}{QP'} = \frac{-QM}{QP} = -\cos A,$$

$$\tan(180^\circ - A) = \tan EQP' = \frac{P'M'}{QM'} = \frac{-PM}{QM} = -\tan A.$$

And similarly the other ratios may be compared.

98. To shew that $\sin(90^\circ + A) = \cos A$, and

$$\cos(90^\circ + A) = -\sin A.$$



Let the angle $EQP = A$.

Draw QP' at right angles to QP, and make $QP' = QP$.

Draw PM and $P'M'$ at right angles to EQW.

Then since the angles $P'QM'$ and PQM make up a right angle, and the angles QPM and PQM make up a right angle,

Angle $P'QM' =$ angle QPM.

Also right angle $P'M'Q =$ right angle PMQ,

and side $P'Q =$ side PQ, opposite equal angles in each;

$\therefore$ the triangles $P'QM'$, QPM are equal in all respects, and

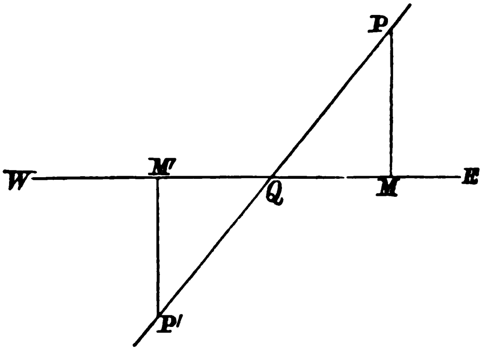
$$P'M' = QM, \text{ and } QM' = PM.$$

Then $\sin(90^\circ + A) = \sin EQP' = \dfrac{P'M'}{QP'} = \dfrac{QM}{QP} = \cos A$,

and $\cos(90^\circ + A) = \cos EQP' = \dfrac{QM'}{QP'} = \dfrac{-PM}{QP} = -\sin A$.

99. To shew that $\sin(180^\circ + A) = -\sin A$, and

$\cos(180^\circ + A) = -\cos A$.



Let the angle $EQP = A$:

Produce EQ to W and PQ to P', making $QP' = QP$.

Draw PM, $P'M'$ at right angles to EW.

Then the angle EQP' measured in the positive direction $= 180^\circ + A$.

The triangles PQM, $P'QM'$ are geometrically equal.

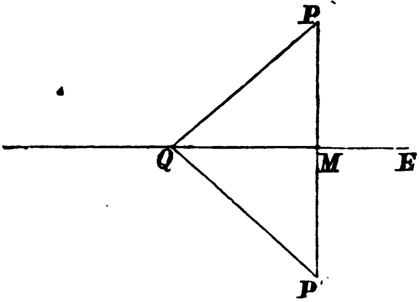
Now $\sin(180^\circ + A) = \sin EQP' = \dfrac{P'M'}{QP'} = \dfrac{-PM}{QP} = -\sin A$,

and $\cos(180^\circ + A) = \cos EQP' = \dfrac{QM'}{QP'} = \dfrac{-QM}{QP} = -\cos A$.

100. To shew that $\sin(-A) = -\sin A$, and $\cos(-A) = \cos A$.

Let the angle $EQP = A$.

Draw PM at right angles to EW and produce PM to P' making $MP' = MP$.



Join QP'.

Then the triangles PQM, $P'QM$ are geometrically equal, and the angle EQP', which is numerically equal to EQP, will, if regarded as measured in a negative direction, be represented by $-A$.

Now $$\sin(-A) = \sin EQP' = \frac{P'M}{QP'} = \frac{-PM}{QP} = -\sin A,$$

and $$\cos(-A) = \cos EQP' = \frac{QM}{QP'} = \frac{QM}{QP} = \cos A.$$

101. To shew that $\sin(360^\circ - A) = -\sin A$, and
$\cos(360^\circ - A) = \cos A$.

Making the same construction as in the preceding Article,

angle EQP' measured in the positive direction $= 360^\circ - A$.

Then $$\sin(360^\circ - A) = \sin EQP' = \frac{P'M}{QP'} = \frac{-PM}{QP} = -\sin A,$$

and $$\cos(360^\circ - A) = \cos EQP' = \frac{QM}{QP'} = \frac{QM}{QP} = \cos A.$$

EXAMPLES.—XXII.

1. Prove the following relations:

(1) $\sec(180^\circ - A) = -\sec A$, (2) $\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$,

(3) $\tan(180^\circ + A) = \tan A$, (4) $\sec(\pi + \theta) = -\sec\theta$,

(5) $\tan(-\theta) = -\tan\theta$, (6) $\cot(2\pi - \theta) = -\cot\theta$.

2. State and prove the relations subsisting between the cosecants of B° and $(90 + B)^\circ$, also of ϕ and $\pi + \phi$.

3. State and prove the relations subsisting between the secants of A° and $(90 + A)^\circ$, also of θ and $\frac{\pi}{2} - \theta$.

102. The relations between the Trigonometrical Ratios of different angles explained in this chapter may be remembered with more ease by the consideration of the following facts.

There are Six Trigonometrical Ratios.

For an angle in the

First Quadrant all the Ratios are Positive,
Second Quadrant all are Negative except the Sine and Cosecant,
Third.. Tangent and Cotangent,
Fourth .... Cosine and Secant.

Thus

(1) $\cos(90^\circ - A) = \sin A$,
$\tan(90^\circ - A) = \cot A$.

(2) $\cos(180^\circ - A) = -\cos A$,
$\sin(180^\circ - A) = \sin A$.

(3) $\sin(180^\circ + A) = -\sin A$,
$\tan(180^\circ + A) = \tan A$.

(4) $\sin(360^\circ - A) = -\sin A$,
$\cos(360^\circ - A) = \cos A$.

EXAMPLES.—XXIII.

Find the values of the following Ratios:

(1) $\sin 120^\circ$, (2) $\cos 120^\circ$, (3) $\sin 135^\circ$, (4) $\cos 135^\circ$,

(5) $\sin 150^\circ$, (6) $\cos 150^\circ$, (7) $\sin 225^\circ$, (8) $\sin 240^\circ$,

(9) $\tan 300^\circ$, (10) $\operatorname{cosec} 300^\circ$, (11) $\sec 315^\circ$, (12) $\cot 330^\circ$.

CHAPTER IX.

ON THE SOLUTION OF TRIGONOMETRICAL EQUATIONS.

103. The Solution of a Trigonometrical Equation is the process of finding what angle an unknown letter representing an angular magnitude must stand for in order that the equation may be true.

(1) Suppose we have to find the value of θ which will satisfy the equation

$$\cos\theta + \sec\theta = \frac{5}{2}.$$

Our first step is to put $\frac{1}{\cos\theta}$ in the place of $\sec\theta$, so that we may have only one function of the unknown angle in the equation: thus

$$\cos\theta + \frac{1}{\cos\theta} = \frac{5}{2}.$$

We then proceed to solve the equation just as we should solve an algebraical equation in which x occupied the place of $\cos\theta$: thus

$$2\cos^2\theta + 2 = 5\cos\theta,$$

$$2\cos^2\theta - 5\cos\theta = -2,$$

$$\cos^2\theta - \frac{5}{2}\cos\theta = -1,$$

$$\cos^2\theta - \frac{5}{2}\cos\theta + \frac{25}{16} = \frac{9}{16},$$

$$\cos\theta - \frac{5}{4} = \pm\frac{3}{4}.$$

$$\cos\theta = 2 \text{ or } \frac{1}{2}.$$

Now the value 2 is inadmissible, for the cosine of every angle is not greater than 1.

The other value $\frac{1}{2}$ is the value of the cosine of 60°. (Art. 76.)

Hence $\cos\theta = \cos 60°$.

That is, *one* value of θ which satisfies the equation is 60°.

We shall explain hereafter our reason for writing the word *one* in italics.

(2) To solve the equation $3\sin\theta = 2\cos^2\theta$.

$$3\sin\theta = 2(1-\sin^2\theta);$$

$$\therefore\ 3\sin\theta = 2 - 2\sin^2\theta,$$

$$\text{or}\quad 2\sin^2\theta + 3\sin\theta = 2,$$

$$\sin^2\theta + \frac{3}{2}\sin\theta = 1,$$

$$\sin^2\theta + \frac{3}{2}\sin\theta + \frac{9}{16} = \frac{25}{16},$$

$$\sin\theta + \frac{3}{4} = \pm\frac{5}{4},$$

$$\sin\theta = \frac{1}{2}\ \text{or}\ -2.$$

The value -2 is inadmissible, for the sine of an angle cannot be numerically greater than 1.

The other value $\frac{1}{2}$ is the value of the sine of 30°. (Art. 74.)

Hence $\sin\theta = \sin 30°$.

That is, *one* value of θ which satisfies the equation is 30°.

EXAMPLES.—XXIV.

Find a value of θ which will satisfy the following equations:

(1) $\sin\theta + \cos\theta = 0$.
(2) $\sin\theta - \cos\theta = 0$.
(3) $\sin\theta = \tan\theta$.
(4) $\cos\theta = \cot\theta$.
(5) $2\sin\theta = \tan\theta$.
(6) $3\sin\theta = 2\cos^2\theta$.
(7) $\sin\theta + \cos^2\theta \,.\, \operatorname{cosec}\theta = 2$.
(8) $\tan\theta = 4 - 3\cot\theta$.
(9) $4\sec^2\theta - 7\tan^2\theta = 3$.
(10) $\cos\theta \,.\, \operatorname{cosec}\theta + \sin\theta \,.\, \sec\theta = \dfrac{4}{\sqrt{3}}$.
(11) $3\sin^2\theta - \cos^2\theta + (\sqrt{3}+1)(1 - 2\sin\theta) = 0$.
(12) $3\cos^2\theta - \sin^2\theta + (\sqrt{3}+1)(1 - 2\cos\theta) = 0$.
(13) $\sec\theta \,.\, \operatorname{cosec}\theta + 2\cot\theta = 4$.
(14) $\sin\theta + \cos\theta = \sqrt{2}$.
(15) $\operatorname{cosec}^4\theta - \sec^4\theta = 2\operatorname{cosec}^3\theta\sec^3\theta$.
(16) $\tan\theta + \cot\theta = 2$.
(17) $\sin\theta - \cos\theta = \sqrt{2}$.
(18) $\cot^2\theta - \tan^2\theta = 2\operatorname{cosec}\theta \,.\, \sec\theta$.
(19) $\sin\theta\sqrt{3} = \sqrt{3} - \cos\theta$.
(20) $\tan^2\theta + 4\sin^2\theta = 3$.

104. We have already stated (Art. 70) that if the value of a Trigonometrical Ratio be given we cannot fix on *one* particular angle to which it exclusively belongs. This statement was confirmed by many of the conclusions at which we arrived in Chap. VIII. For instance, since

$$\sin(180^\circ - A) = \sin A,$$

it follows that the sines of the angles A and $180^\circ - A$ have the same value, that is, if we know that $\sin\theta = \frac{1}{2}$, θ may have either of *two* values, one of which is 30° and the other 150°.

Now we know that

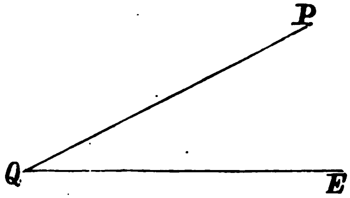
$$\sin(180^\circ - A) = \sin A \text{ and } \operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A,$$

$$\cos(360^\circ - A) = \cos A \text{ and } \sec(360^\circ - A) = \sec A,$$

$$\tan(180^\circ + A) = \tan A \text{ and } \cot(180^\circ + A) = \cot A.$$

Thus for each given value of any one of the Trigonometrical Ratios there are *two* angles and two only between 0° and 360° for which that Ratio is the same in magnitude and sign.

105. Suppose QE to be the primitive line, and QP the revolving line, and let the angle EQP, less than 360°, be denoted by A.



Now suppose QP to make a *complete* revolution, that is, to start from the position indicated in the diagram and to revolve till it comes back to that position.

Then the revolving line will have described an angle $360^\circ + A$.

Our triangle of reference will then be the same for an angle $360^\circ + A$ as for an angle A.

Hence, $\sin(360^\circ + A) = \sin A$,

and $\cos(360^\circ + A) = \cos A$.

And the same holds good for the other Ratios.

Hence, expressing ourselves for brevity in the symbols of the circular system, if a be the circular measure of an angle for which any one of the trigonometrical ratios has an assigned value,

$2\pi + a$ will represent an angle for which the value of that particular ratio is the same.

Now let the revolving line make a second complete revolution, then it will have described an angle

$$2\pi + 2\pi + a, \text{ or, } 4\pi + a.$$

And so $4\pi + a$ will represent an angle for which the value of the above-mentioned ratio will be the same.

And, generally, if the revolving line, after having traced out the angle a makes n revolutions,

$2n\pi + a$ will represent an angle for which the value of any particular ratio is the same as it is for a.

Now since n may have any integral value from 1 to ∞ there will be an infinite number of angles for which the value of any one of the Trigonometrical ratios is the same as it is for the angle a.

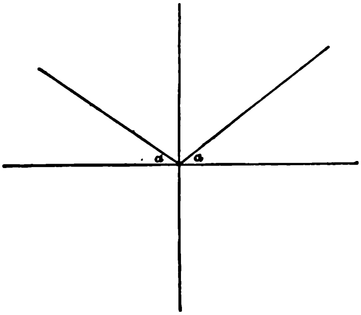
Again, if a be the circular measure of an angle traced out by a line revolving in the positive direction, $-(2\pi - a)$ will be the measure of an angle traced out by the line revolving in the *negative* direction, for which the triangle of reference will be the same as for the positive angle a.

If the line then make n complete revolutions in the negative direction, $-2n\pi - (2\pi - a)$ will represent an angle for which the value of any particular ratio is the same as it is for a.

We can now explain the way in which general expressions are found for all angles which have a given trigonometrical ratio.

106. To find a general expression for all angles which have a given sine.

Let a be an angle whose sine is given.



First, reckoning in the positive direction,

$$a \text{ and } \pi - a$$

are angles with the same sine. (Art. 97.)

$$\left.\begin{array}{ll}\text{Also} & 2n\pi + a \\ \text{and} & 2n\pi + (\pi - a)\end{array}\right\}\dots\dots\dots\dots(1)$$

are angles with the same sine. (Art. 105.)

Secondly, reckoning in the negative direction,

$$-(2\pi - a) \text{ and } -\{2\pi - (\pi - a)\},$$

that is, $\quad -(2\pi - a)$ and $-(\pi + a)$,

are angles with the same sine.

$$\left.\begin{array}{ll}\text{Also} & -2n\pi - (2\pi - a) \\ \text{and} & -2n\pi - (\pi + a)\end{array}\right\}\dots\dots\dots\dots(2)$$

are angles with the same sine, n being any positive integer.

Now the angles in (1) and (2) may be arranged thus:

$$2n\pi + a,\quad (2n + 1)\,\pi - a,\quad -(2n + 2)\,\pi + a,\quad -(2n + 1)\,\pi - a,$$

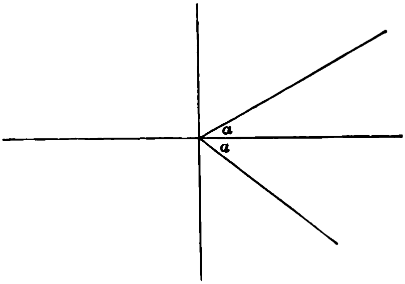
all of which, and no others, are included in the formula

$$n\pi + (-1)^n . a,$$

where n is zero or any positive or negative integer, which is therefore the general expression for all angles which have a given sine.

107. To find a general expression for all angles which have a given cosine.

Let a be an angle whose cosine is given.



First, reckoning in the positive direction,

$$a \text{ and } 2\pi - a$$

are angles with the same cosine. (Art. 101.)

$$\left.\begin{array}{ll}\text{Also} & 2n\pi + a \\ \text{and} & 2n\pi + (2\pi - a)\end{array}\right\}\cdots\cdots(1)$$

are angles with the same cosine. (Art. 105.)

Secondly, reckoning in a negative direction,

$$-(2\pi - a) \text{ and } -a$$

are angles with the same cosine.

$$\left.\begin{array}{ll}\text{Also} & -2n\pi - (2\pi - a) \\ \text{and} & -2n\pi - a\end{array}\right\}\cdots\cdots(2)$$

are angles with the same cosine, n being any positive integer.

Now the angles in (1) and (2) may be arranged thus:

$$2n\pi + a,\ (2n+2)\pi - a,\ -(2n+2)\pi + a,\ -2n\pi - a,$$

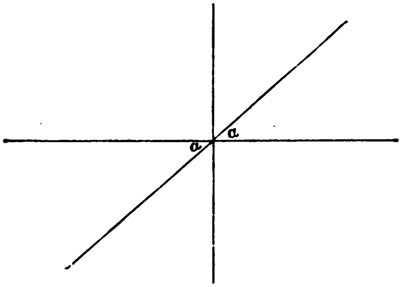
all of which, and no others, are included in the formula

$$2n\pi \pm a,$$

which is therefore the general expression for all angles which have a given cosine.

108. To find a general expression for all angles which have a given tangent.

Let a be an angle whose tangent is given.



First, reckoning in the positive direction,

$$a \text{ and } \pi + a$$

are angles with the same tangent. (Art. 104.)

$$\left.\begin{array}{ll}\text{Also} & 2n\pi + a \\ \text{and} & 2n\pi + (\pi + a)\end{array}\right\} \cdots\cdots\cdots\cdots (1)$$

are angles with the same tangent. (Art. 105.)

Secondly, reckoning in the negative direction,

$$-(2\pi - a) \text{ and } -(\pi - a)$$

are angles with the same tangent.

$$\left.\begin{array}{ll}\text{Also} & -2n\pi - (2\pi - a) \\ \text{and} & -2n\pi - (\pi - a)\end{array}\right\} \cdots\cdots\cdots\cdots (2)$$

are angles with the same tangent, n being any positive integer.

Now the angles in (1) and (2) may be arranged thus:

$$2n\pi + a,\ (2n+1)\pi + a,\ -(2n+2)\pi + a,\ -(2n+1)\pi + a,$$

all of which, and no others, are included in the formula

$$n\pi + a,$$

which is therefore the general expression for all angles which have a given tangent.

109. We shall now explain how to express the Trigonometrical Ratios of any angle in terms of the ratios of a positive angle less than a right angle.

First, when the given angle is *positive.*

If the angle is greater than 360°, subtract from it 360° or any multiple of 360°, and the ratios for the resulting angle are the same as for the original angle.

Thus we obtain an angle less than 360°, and if this angle be greater than 180°, we may subtract 180° from it, and the ratios for the resulting angle will be the same in magnitude, but the signs of all but the tangent and cotangent will be changed. (Art. 102.)

Thus we obtain an angle less than 180°, and if this angle be greater than 90°, we may replace it by its supplement, and the ratios for the resulting angle will be the same in magnitude, but the signs of all but the sine and cosecant will be changed. (Art. 102.)

Thus

$\sin 675^\circ = \sin (360^\circ + 315^\circ) = \sin 315^\circ = -\sin 135^\circ = -\sin 45^\circ$,

$\tan 960^\circ = \tan (720^\circ + 240^\circ) = \tan 240^\circ = \tan 60^\circ$.

Secondly, when the angle is *negative.*

Add 360° or any multiple of 360° so as to obtain a positive angle, for which the ratios will be the same as for the original angle, and then proceed as before.

If the given angle be less than 180°, apply the formulæ obtained from Art. 100.

Ex. $\sin (-825^\circ) = \sin (1080^\circ - 825^\circ) = \sin 255^\circ = -\sin 75^\circ$,

$\tan (-135^\circ) = -\tan 135^\circ = -\tan 45^\circ$.

EXAMPLES.—XXV.

Find the values of the following ratios:

(1) $\sin 480^\circ$. (2) $\cos 480^\circ$. (3) $\sin 495^\circ$.

(4) $\cos 495^\circ$. (5) $\sin 870^\circ$. (6) $\cos 870^\circ$.

(7) $\sin 945^\circ$.
(8) $\sin 960^\circ$.
(9) $\tan 1020^\circ$.
(10) $\operatorname{cosec} 1380^\circ$.
(11) $\sec 1395^\circ$.
(12) $\cot 1410^\circ$.
(13) $\cos 420^\circ$.
(14) $\sec 750^\circ$.
(15) $\tan 945^\circ$.
(16) $\sin 1200^\circ$.
(17) $\sin 1485^\circ$.
(18) $\cos 1470^\circ$.
(19) $\sin 7\pi$.
(20) $\sec 8\pi$.
(21) $\operatorname{cosec} 930^\circ$.
(22) $\cot 1140^\circ$.
(23) $\tan 1305^\circ$.
(24) $\operatorname{cosec} 1740^\circ$.
(25) $\sin(-240^\circ)$.
(26) $\cot(-675^\circ)$.
(27) $\sec(-135^\circ)$.
(28) $\tan(-225^\circ)$.

Examples.—XXVI.

Write down the general value of θ which satisfies the following equations:

(1) $\sin\theta = 1$.
(2) $\cos\theta = 1$.
(3) $\sin\theta = \frac{1}{\sqrt{2}}$.
(4) $\tan\theta = \sqrt{3}$.
(5) $3\sin\theta = 2\cos^2\theta$.
(6) $2\sin\theta = \tan\theta$.
(7) $\tan^2\theta + 4\sin^2\theta = 3$.
(8) $\cos^2\theta = \sin^2\theta$.
(9) $\tan\theta = 4 - 3\cot\theta$.
(10) $\sec^2\theta - \frac{5}{2}\sec\theta + 1 = 0$.

110. The symbol $\sin^{-1}x$ denotes an angle whose sine is x,
$\cos^{-1}x$ .... cosine is x,
and a similar notation is used for the other ratios.

Hence if we know that, for instance, $\tan\theta = \frac{2}{3}$, we may write the general value of θ thus:

$$\theta = n\pi + \tan^{-1}\frac{2}{3}.$$

CHAPTER X.

ON THE TRIGONOMETRICAL RATIOS OF TWO ANGLES.

111. We now proceed to explain the Trigonometrical Functions of the Sum and Difference of two angles. These functions are the most important in the subject, and the student will find that his subsequent progress will depend much on the way in which he has read this Chapter.

112. We shall first establish the following formulæ:

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

$$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B,$$

$$\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B,$$

$$\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B;$$

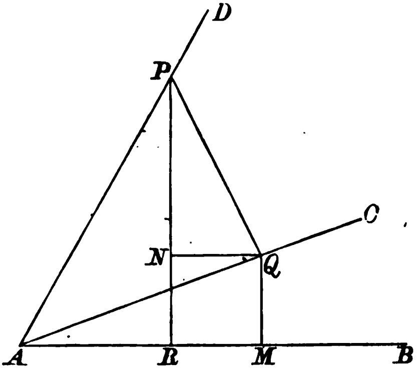
by means of which we can express the sine and cosine of the sum or difference of two angles in terms of the sines and cosines of the angles themselves.

The *diagrams* which we shall employ are only applicable to the cases in which A and B are both positive and less than 90°, also, when we are considering the sum of the angles, $A+B$ is less than 90°, and when we are considering the difference of the angles, A is greater than B. The *formulæ* are however true for all values of A and B. Particular cases may be proved by special constructions of the diagrams, but it is beyond the scope of this treatise to enter into detail on this and similar points.

113. To shew that

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

and $$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$



Let the angle BAC be represented by A and CAD by B.

Then the angle BAD will be represented by $A+B$.

From P, any point in AD, draw PR at right angles to AB, and PQ at right angles to AC.

From Q draw QM at right angles to AB, and QN at right angles to PR.

Then angle $QPN = 90° - PQN = NQA = QAM = A$,

Now $$\sin(A+B) = \sin PAR = \frac{RP}{AP}$$

$$= \frac{RN+NP}{AP} = \frac{QM+NP}{AP} = \frac{QM}{AP} + \frac{NP}{AP}$$

$$= \frac{QM}{AQ} \cdot \frac{AQ}{AP} + \frac{NP}{PQ} \cdot \frac{PQ}{AP}$$

$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B.$$

$$\cos(A+B) = \cos PAR = \frac{AR}{AP}$$

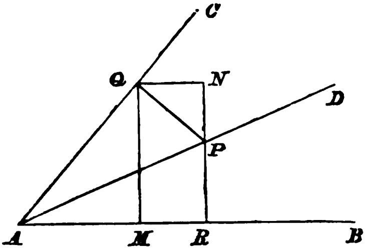
$$= \frac{AM-MR}{AP} = \frac{AM-NQ}{AP} = \frac{AM}{AP} - \frac{NQ}{AP}$$

$$= \frac{AM}{AQ} \cdot \frac{AQ}{AP} - \frac{NQ}{PQ} \cdot \frac{PQ}{AP}$$

$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$

114. To shew that $\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B$,

and $\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B$.



Let the angle BAC be represented by A and CAD by B.

Then the angle BAD will be represented by $A-B$.

From P, any point in AD, draw PR at right angles to AB and PQ at right angles to AC.

From Q draw QM at right angles to AB and QN at right angles to RP produced.

Now $QPN = 90^\circ - PQN = CQN = BAC = A$.

$$\text{Then } \sin(A-B) = \sin PAR = \frac{RP}{AP}$$

$$= \frac{RN - NP}{AP} = \frac{QM - NP}{AP} = \frac{QM}{AP} - \frac{NP}{AP}$$

$$= \frac{QM}{AQ} \cdot \frac{AQ}{AP} - \frac{NP}{PQ} \cdot \frac{PQ}{AP}$$

$$= \sin A \,.\, \cos B - \cos A \,.\, \sin B.$$

$$\cos(A-B) = \cos PAR = \frac{AR}{AP}$$

$$= \frac{AM + MR}{AP} = \frac{AM + NQ}{AP} = \frac{AM}{AP} + \frac{NQ}{AP}$$

$$= \frac{AM}{AQ} \cdot \frac{AQ}{AP} + \frac{NQ}{PQ} \cdot \frac{PQ}{AP}$$

$$= \cos A \,.\, \cos B + \sin A \,.\, \sin B.$$

6—2

115. We shall now give some important examples of the application of the formulæ which we have established.

Ex. 1. To find the value of sin 75°.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ . \cos 30^\circ + \cos 45^\circ . \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2} \quad \text{(Arts. 74, 75)}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

Ex. 2. To find the value of cos 15°.

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ . \cos 30^\circ + \sin 45^\circ . \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}},\end{aligned}$$

a result identical with the value of sin 75°, in accordance with Art. 96, for

$$\sin 75^\circ = \cos(90^\circ - 75^\circ) = \cos 15^\circ.$$

Ex. 3. To shew that $\sin(90^\circ + A) = \cos A$.

$$\begin{aligned}\sin(90^\circ + A) &= \sin 90^\circ . \cos A + \cos 90^\circ . \sin A\\ &= 1 . \cos A + 0 . \sin A \quad \text{(Art. 77)}\\ &= \cos A.\end{aligned}$$

And similarly other relations between trigonometrical functions established in Chapter VIII. may be proved.

Ex. 4. To find a value of θ which satisfies the equation

$$\sin\theta + \cos\theta = 0.$$

Multiply both sides by $\frac{1}{\sqrt{2}}$.

Then $$\sin\theta\,.\,\frac{1}{\sqrt{2}}+\cos\theta\,.\,\frac{1}{\sqrt{2}}=0\,;$$

$$\therefore\ \sin\theta\,.\cos 45^\circ+\cos\theta\,.\sin 45^\circ=0,$$

or, $$\sin(\theta+45^\circ)=0\,;$$

$$\therefore\ \theta+45^\circ=0\,;$$

$$\therefore\ \theta=-45^\circ.$$

EXAMPLES.—XXVII.

Prove the following relations:

(1) $\sin(A+B)\,.\sin(A-B)=\sin^2 A-\sin^2 B.$

(2) $\sin(\alpha+\beta)\,.\sin(\alpha-\beta)=\cos^2\beta-\cos^2\alpha.$

(3) $2\sin(x+y)\,.\cos(x-y)=\sin 2x+\sin 2y.$

(4) $2\cos(x+y)\,.\sin(x-y)=\sin 2x-\sin 2y.$

(5) $\cos(A+B)\,.\cos(A-B)=\cos^2 A-\sin^2 B.$

(6) $\cos(\alpha+\beta)\,.\cos(\alpha-\beta)=\cos^2\beta-\sin^2\alpha.$

(7) $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\,.\cos B}.$

(8) $\tan\alpha-\tan\beta=\dfrac{\sin(\alpha-\beta)}{\cos\alpha\,.\cos\beta}.$

(9) $\tan^2 B-\tan^2 C=\dfrac{\sin(B+C)\,.\sin(B-C)}{\cos^2 B\,.\cos^2 C}.$

(10) $\cot^2\alpha-\cot^2\beta=\dfrac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\,.\sin^2\beta}.$

EXAMPLES.—XXVIII.

(1) Shew that $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(2) Shew that $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(3) Shew that $\tan 75^\circ = 2 + \sqrt{3}$.

(4) Shew that $\cot 75^\circ = 2 - \sqrt{3}$.

(5) If $\sin \alpha = \dfrac{1}{3}$ and $\sin \beta = \dfrac{2}{3}$ find the value of $\sin(\alpha + \beta)$.

(6) If $\cos \alpha = \dfrac{3}{4}$ and $\cos \beta = \dfrac{2}{5}$ find the value of $\sin(\alpha - \beta)$.

(7) If $\sin \alpha = \cdot\dot{5}$ and $\cos \beta = \dfrac{1}{\sqrt{2}}$ find the value of $\cos(\alpha + \beta)$.

(8) If $\cos \alpha = \cdot 0\dot{3}$ and $\sin \beta = \dfrac{1}{2}$ find the value of $\cos(\alpha - \beta)$.

EXAMPLES.—XXIX.

Apply the formulæ established in this chapter to prove the following relations between the Trigonometrical Functions of angles.

(1) $\cos(90^\circ + A) = -\sin A$.

(2) $\sin(180^\circ + A) = -\sin A$.

(3) $\cos(\pi + \theta) = -\cos\theta$.

(4) $\sin\left(\dfrac{3\pi}{2} + \theta\right) = -\cos\theta$.

(5) $\operatorname{cosec}\left(\dfrac{\pi}{2} + \alpha\right) = \sec\alpha$.

(6) $\tan(\pi + \alpha) = \tan\alpha$.

(7) $\sin(2\pi - \theta) = -\sin\theta$.

(8) $\tan(2\pi - \theta) = -\tan\theta$.

(9) $\sec(180^\circ - \theta) = -\sec\theta$.

(10) $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$.

EXAMPLES.—XXX.

Find a value of θ to satisfy the following equations, by a process similar to that given in Art. 115.

(1) $\sin\theta - \cos\theta = 0.$ (2) $\sin\theta + \cos\theta = 1.$

(3) $\sin\theta - \cos\theta = \sqrt{\frac{3}{2}}.$ (4) $\sin\theta + \cos\theta = \frac{\sqrt{3}+1}{2}.$

(5) $\sin\theta + \cos\theta = \sqrt{2}.$ (6) $\sin\theta - \cos\theta = \frac{\sqrt{3}-1}{2}.$

116. Collecting the formulæ of Arts. 113, 114, we next arrange them thus:

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

$$\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B,$$

$$\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B,$$

$$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$

Hence, by addition and subtraction, we obtain the following:

$$\sin(A+B) + \sin(A-B) = 2\sin A \,.\, \cos B,$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \,.\, \sin B,$$

$$\cos(A-B) + \cos(A+B) = 2\cos A \,.\, \cos B,$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \,.\, \sin B.$$

Now let $A+B=P,$

and $A-B=Q.$

Then $2A = P+Q,$ and $2B = P-Q;$

$$\therefore\ A = \frac{P+Q}{2}, \text{ and } B = \frac{P-Q}{2};$$

So that the formulæ may be put in this form:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\,.\,\cos\frac{P-Q}{2},$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\,.\,\sin\frac{P-Q}{2},$$

$$\cos Q + \cos P = 2\cos\frac{P+Q}{2}\,.\,\cos\frac{P-Q}{2},$$

$$\cos Q - \cos P = 2\sin\frac{P+Q}{2}\,.\,\sin\frac{P-Q}{2}.$$

117. As these results are of very great importance we shall repeat them separately, explaining each in words.

(1) $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\,.\,\cos\frac{P-Q}{2}$,

that is, *the sum of the sines of two angles is equal to twice the product of the sine of half the sum of the angles into the cosine of half their difference.*

Ex. $\sin 10\theta + \sin 6\theta = 2\sin\frac{10\theta+6\theta}{2}\,.\,\cos\frac{10\theta-6\theta}{2}$

$$= 2\sin 8\theta\,.\,\cos 2\theta.$$

(2) $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\,.\,\sin\frac{P-Q}{2}$,

that is, *the difference of the sines of two angles is equal to twice the product of the cosine of half the sum of the angles into the sine of half their difference.*

Ex. $\sin 8a - \sin 4a = 2\cos\frac{8a+4a}{2}\,.\,\sin\frac{8a-4a}{2}$

$$= 2\cos 6a\,.\,\sin 2a.$$

(3) $$\cos Q+\cos P=2\cos\frac{P+Q}{2}.\cos\frac{P-Q}{2},$$

that is, *the sum of the cosines of two angles is equal to twice the product of the cosine of half the sum of the angles into the cosine of half their difference.*

Ex. $$\cos\theta+\cos 3\theta=2\cos\frac{3\theta+\theta}{2}.\cos\frac{3\theta-\theta}{2}$$
$$=2\cos 2\theta.\cos\theta.$$

(4) $$\cos Q-\cos P=2\sin\frac{P+Q}{2}.\sin\frac{P-Q}{2},$$

that is, *the difference of the cosines of two angles is equal to twice the product of the sine of half the sum of the angles into the sine of half their difference.*

Ex. $$\cos 3a-\cos 7a=2\sin\frac{7a+3a}{2}.\sin\frac{7a-3a}{2}$$
$$=2\sin 5a.\sin 2a.$$

118. As the formulæ at the end of Art. 116 teach us how to replace the sum or difference of two sines or cosines by the product of two sines or cosines, so the formulæ at the beginning of Art. 116, when read from right to left, thus:

$$2\sin A.\cos B=\sin(A+B)+\sin(A-B),$$
$$2\cos A.\sin B=\sin(A+B)-\sin(A-B),$$
$$2\cos A.\cos B=\cos(A-B)+\cos(A+B),$$
$$2\sin A.\sin B=\cos(A-B)-\cos(A+B),$$

furnish rules for replacing the product of two sines or cosines by the sum or difference of sines or cosines.

For example,

$$\sin 5\theta.\cos 3\theta=\frac{1}{2}\{\sin(5\theta+3\theta)+\sin(5\theta-3\theta)\}$$
$$=\frac{1}{2}(\sin 8\theta+\sin 2\theta),$$

$$\sin\theta.\sin 3\theta=\frac{1}{2}\{\cos(3\theta-\theta)-\cos(3\theta+\theta)\}$$
$$=\frac{1}{2}(\cos 2\theta-\cos 4\theta).$$

EXAMPLES.—XXXI.

Prove the following relations :

(1) $\sin 6A + \sin 4A = 2 \sin 5A \,.\, \cos A.$

(2) $\sin 5A - \sin 3A = 2 \cos 4A \,.\, \sin A.$

(3) $\cos 7\theta + \cos 9\theta = 2 \cos 8\theta \,.\, \cos \theta.$

(4) $\cos \theta - \cos 5\theta = 2 \sin 3\theta \,.\, \sin 2\theta.$

(5) $\sin \alpha + \sin 4\alpha = 2 \sin \frac{5\alpha}{2} \,.\, \cos \frac{3\alpha}{2}.$

(6) $\cos 5\alpha - \cos 8\alpha = 2 \sin \frac{13\alpha}{2} \,.\, \sin \frac{3\alpha}{2}.$

(7) $2 \sin 5\theta \,.\, \cos 7\theta = \sin 12\theta - \sin 2\theta.$

(8) $2 \sin 3\theta \,.\, \sin 5\theta = \cos 2\theta - \cos 8\theta.$

(9) $2 \cos \alpha \,.\, \cos 4\alpha = \cos 5\alpha + \cos 3\alpha.$

(10) $2 \cos \alpha \,.\, \sin 2\alpha = \sin 3\alpha + \sin \alpha.$

(11) $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}.$

(12) $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A.$

(13) $\frac{\sin 2A + \sin A}{\cos 2A + \cos A} = \tan \frac{3A}{2}.$

(14) $\cos (30^\circ - \theta) - \cos (30^\circ + \theta) = \sin \theta.$

(15) $\cos \left(\frac{\pi}{3} + \theta\right) + \cos \left(\frac{\pi}{3} - \theta\right) = \cos \theta.$

(16) $\sin \left(\frac{\pi}{3} + \alpha\right) - \sin \left(\frac{\pi}{3} - \alpha\right) = \sin \alpha.$

(17) $\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \frac{\alpha + \beta}{2}.$

(18) $\dfrac{\sin\alpha - \sin\beta}{\cos\beta + \cos\alpha} = \tan\dfrac{\alpha-\beta}{2}$.

(19) $\dfrac{\sin 5\theta + \sin 3\theta}{\cos 3\theta - \sin 5\theta} = \cot\theta$.

(20) $\dfrac{\cos\alpha + \cos\beta}{\cos\beta - \cos\alpha} = \dfrac{\cot\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$.

119. We can also express the sum or difference of a sine and a cosine as the product of sines or cosines.

For since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ (Art. 96),

$$\sin\alpha + \cos\theta = \sin\alpha + \sin\left(\frac{\pi}{2} - \theta\right)$$

$$= 2\sin\frac{1}{2}\left(\alpha + \frac{\pi}{2} - \theta\right).\cos\frac{1}{2}\left(\alpha - \frac{\pi}{2} + \theta\right).$$

Again

$$\sin 40° + \cos 60° = \sin 40° + \sin 30°$$

$$= 2\sin 35°.\cos 5°.$$

Examples.—XXXII.

Express as the product of sines or cosines:

(1) $\sin\alpha - \cos\beta$, (2) $\sin\left(\frac{\pi}{2} + \alpha\right) + \cos\left(\frac{\pi}{2} - \alpha\right)$,

(3) $\sin\alpha + \cos\alpha$, (4) $\sin\alpha - \cos\alpha$,

(5) $\sin 30° + \cos 80°$, (6) $\sin 20° - \cos 80°$,

(7) $\sin\frac{\pi}{4} + \cos\frac{\pi}{6}$, (8) $\sin\frac{\pi}{3} - \cos\frac{\pi}{5}$.

120. We now proceed to explain how the Tangent of the sum and difference of two angles can be expressed in terms of the tangents of the angles themselves.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A.\cos B+\cos A.\sin B}{\cos A.\cos B-\sin A.\sin B},$$

and, dividing each term of the numerator and denominator by $\cos A.\cos B$,

$$=\frac{\dfrac{\sin A.\cos B}{\cos A.\cos B}+\dfrac{\cos A.\sin B}{\cos A.\cos B}}{\dfrac{\cos A.\cos B}{\cos A.\cos B}-\dfrac{\sin A.\sin B}{\cos A.\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A.\tan B}.$$

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$

$$=\frac{\sin A.\cos B-\cos A.\sin B}{\cos A.\cos B+\sin A.\sin B}$$

$$=\frac{\dfrac{\sin A.\cos B}{\cos A.\cos B}-\dfrac{\cos A.\sin B}{\cos A.\cos B}}{\dfrac{\cos A.\cos B}{\cos A.\cos B}+\dfrac{\sin A.\sin B}{\cos A.\cos B}}$$

$$=\frac{\tan A-\tan B}{1+\tan A.\tan B}.$$

COR. We proved in Art. 75 that $\tan 45^\circ=1$.

Hence

$$\tan(45^\circ+A)=\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ.\tan A}$$

$$=\frac{1+\tan A}{1-\tan A},$$

$$\tan(45^\circ-A)=\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ.\tan A}$$

$$=\frac{1-\tan A}{1+\tan A}.$$

121. The results of the preceding article may be obtained without assuming the formulæ for the sine and cosine, thus:

Taking the Diagram of Art. 113, we have

$$\tan(A+B) = \frac{PR}{AR} = \frac{NR+PN}{AM-MR}$$

$$= \frac{QM+PN}{AM-NQ}$$

$$= \frac{\dfrac{QM}{AM} + \dfrac{PN}{AM}}{1 - \dfrac{NQ}{AM}}.$$

Now $\dfrac{QM}{AM} = \tan A$,

and observing that PNQ, MQA are similar triangles,

$$\frac{PN}{AM} = \frac{PQ}{AQ} = \tan B,$$

$$\frac{NQ}{AM} = \frac{NQ}{PN} \cdot \frac{PN}{AM} = \tan A \,.\, \tan B;$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}.$$

122. Again, taking the diagram of Art. 114, we have

$$\tan(A-B) = \frac{PR}{AR} = \frac{NR-PN}{AM+MR}$$

$$= \frac{QM-PN}{AM+NQ}$$

$$= \frac{\dfrac{QM}{AM} - \dfrac{PN}{AM}}{1 + \dfrac{NQ}{AM}}.$$

Now $\dfrac{QM}{AM} = \tan A$,

and observing that *PNQ*, *MQA* are similar triangles,

$$\frac{PN}{AM} = \frac{PQ}{AQ} = \tan B,$$

$$\frac{NQ}{AM} = \frac{NQ}{PN} \cdot \frac{PN}{AM} = \tan A \,.\, \tan B\,;$$

$$\therefore\ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}.$$

EXAMPLES.—XXXIII.

(1) $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha \,.\, \tan\beta.$

(2) $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha - \tan\beta} = \tan\alpha \,.\, \tan(\alpha + \beta).$

(3) $\dfrac{\tan\alpha - \tan\beta}{\cot\alpha + \tan\beta} = \tan\alpha \,.\, \tan(\alpha - \beta).$

(4) $\tan\dfrac{\phi + \psi}{2} + \tan\dfrac{\phi - \psi}{2} = \dfrac{2\sin\phi}{\cos\phi + \cos\psi}.$

(5) $\sin\phi = \sin\psi \,.\, \cos(\phi - \psi) + \cos\psi \,.\, \sin(\phi - \psi).$

(6) $\cos\phi = \sin\psi \,.\, \sin(\phi + \psi) + \cos\psi \,.\, \cos(\phi + \psi).$

(7) $(\cos\alpha + \cos\beta)\{1 - \cos(\alpha + \beta)\} = (\sin\alpha + \sin\beta) \,.\, \sin(\alpha + \beta).$

(8) $\dfrac{\sin(\alpha + \beta)}{\sin\alpha + \sin\beta} = \dfrac{\cos\dfrac{\alpha + \beta}{2}}{\cos\dfrac{\alpha - \beta}{2}}.$

(9) $\dfrac{\sin(\alpha + \beta)}{\sin\alpha - \sin\beta} = \dfrac{\sin\dfrac{\alpha + \beta}{2}}{\sin\dfrac{\alpha - \beta}{2}}.$

(10) $\tan\dfrac{\alpha + \beta}{2} + \tan\dfrac{\alpha - \beta}{2} = \dfrac{2\sin\alpha}{\cos\alpha + \cos\beta}.$

(11) $\tan\frac{\alpha+\beta}{2} - \tan\frac{\alpha-\beta}{2} = \frac{2\sin\beta}{\cos\alpha + \cos\beta}$.

(12) $\frac{\cos\alpha - \cos\beta}{\sin\alpha + \sin\beta} = \tan\frac{\beta-\alpha}{2}$.

(13) $\cot\beta - \tan\alpha = \frac{\cos(\alpha+\beta)}{\cos\alpha \,.\, \sin\beta}$.

(14) $\cot\theta + \tan\phi = \frac{\cos(\phi-\theta)}{\cos\phi \,.\, \sin\theta}$.

(15) $\tan^2\alpha - \tan^2\beta = \frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha \,.\, \cos^2\beta}$.

(16) $1 + \tan\alpha \,.\, \tan\beta = \frac{\cos(\alpha-\beta)}{\cos\alpha \,.\, \cos\beta}$.

(17) $1 - \tan\alpha \,.\, \tan\beta = \frac{\cos(\alpha+\beta)}{\cos\alpha \,.\, \cos\beta}$.

(18) $\frac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = \cot\alpha \,.\, \tan\beta$.

(19) $\frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \,.\, \tan^2 y} = \tan(x+y) \,.\, \tan(x-y)$.

(20) $\cot(\theta + 45^\circ) = \frac{\cot\theta - 1}{\cot\theta + 1}$.

(21) $\sin\theta + \cos\theta = \sqrt{2} \,.\, \sin(45^\circ + \theta)$.

(22) $\cos\theta - \sin\theta = \sqrt{2} \,.\, \sin\left(\frac{\pi}{4} - \theta\right)$.

(23) $\frac{\tan\alpha - \tan\beta}{\tan\alpha + \tan\beta} = \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$.

(24) $\frac{\cot x - \cot y}{\cot x + \cot y} = \frac{\sin(y-x)}{\sin(y+x)}$.

(25) $\cos(A-B) + \sin(A+B) = 2\sin(A+45^\circ) \,.\, \cos(B-45^\circ)$.

(26) $\cos(A-B) - \sin(A+B) = 2\sin(45^\circ - A) \,.\, \cos(45^\circ + B)$.

(27) $\cos(A+B) + \sin(A-B) = 2\sin(45^\circ + A) \,.\, \cos(45^\circ + B)$.

(28) $\cos(A+B) - \sin(A-B) = 2\sin(45^0 - A) \, . \cos(45^0 - B).$

(29) $\dfrac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\dfrac{\cot\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$

(30) $\sec 72^0 - \sec 36^0 = \sec 60^0.$

(31) $\sin 108^0 = (\sin 81^0 + \sin 9^0)(\sin 81^0 - \sin 9^0).$

(32) $\dfrac{\cos 3^0 - \cos 33^0}{\sin 3^0 + \sin 33^0} = \tan 15^0.$

(33) $\dfrac{\sin 33^0 + \sin 3^0}{\cos 33^0 + \cos 3^0} = \tan 18^0.$

(34) $\dfrac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0} = \tan 54^0.$

(35) $\dfrac{\cos 27^0 - \sin 27^0}{\cos 27^0 + \sin 27^0} = \tan 18^0.$

(36) $\tan 50^0 + \cot 50^0 = 2\sec 10^0.$

CHAPTER XI.

ON THE TRIGONOMETRICAL RATIOS FOR MULTIPLE AND SUBMULTIPLE ANGLES.

123. THE angles $2A$, $3A$, $4A$ ...... are called Multiples of A, and the angles $\frac{A}{2}$, $\frac{A}{3}$, $\frac{A}{4}$ ...... are called Submultiples of A.

We shall shew how to express the Trigonometrical Ratios of $2A$, $3A$ and $\frac{A}{2}$ in terms of the ratios of A.

124.
$$\begin{aligned}\sin 2A &= \sin(A+A)\\ &= \sin A \,.\, \cos A + \cos A \,.\, \sin A\\ &= \sin A \,.\, \cos A + \sin A \,.\, \cos A\\ &= 2\sin A \,.\, \cos A.\end{aligned}$$

$$\begin{aligned}\text{Cos } 2A &= \cos(A+A)\\ &= \cos A \,.\, \cos A - \sin A \,.\, \sin A\\ &= \cos^2 A - \sin^2 A.\end{aligned}$$

Now we may put $1-\sin^2 A$ in the place of $\cos^2 A$, (Art. 84) and we then have

$$\begin{aligned}\cos 2A &= 1-\sin^2 A - \sin^2 A\\ &= 1-2\sin^2 A.\end{aligned}$$

Or we may put $1-\cos^2 A$ in the place of $\sin^2 A$, (Art. 84) and we then have

$$\begin{aligned}\cos 2A &= \cos^2 A - (1-\cos^2 A)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1.\end{aligned}$$

$$\tan 2A = \tan (A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A}, \text{ (Art. 120)}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

125. If we put A in the place of $2A$, and $\frac{A}{2}$ in the place of A, we have

$$\sin A = 2 \sin \frac{A}{2} \cdot \cos \frac{A}{2},$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 1 - 2 \sin^2 \frac{A}{2}$$

$$= 2 \cos^2 \frac{A}{2} - 1,$$

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}.$$

Hence we can shew that $\frac{1 - \cos A}{1 + \cos A} = \tan^2 \frac{A}{2}$, a formula of great importance.

For
$$\frac{1 - \cos A}{1 + \cos A} = \frac{1 - \left(1 - 2 \sin^2 \frac{A}{2}\right)}{1 + \left(2 \cos^2 \frac{A}{2} - 1\right)} = \frac{2 \sin^2 \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \tan^2 \frac{A}{2}.$$

EXAMPLES.—XXXIV.

Prove the following relations:

(1) $\sin 2A = \frac{2 \cot A}{1 + \cot^2 A}$.

(2) $\dfrac{\sin 2A}{1+\cos 2A} \cdot \dfrac{\cos A}{1+\cos A} = \tan\dfrac{A}{2}$.

(3) $\operatorname{cosec} A + \cot A = \cot\dfrac{A}{2}$.

(4) $\tan\theta + \cot\theta = \dfrac{2}{\sin 2\theta}$.

(5) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$.

(6) $2\operatorname{cosec} 2A = \sec A \,.\, \operatorname{cosec} A$.

(7) $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

(8) $\sec^2\theta = \dfrac{2\sec 2\theta}{1+\sec 2\theta}$.

(9) $\dfrac{1-2\sin^2 A}{1+\sin 2A} = \dfrac{1-\tan A}{1+\tan A}$.

(10) $\cot\theta - 2\cot 2\theta = \tan\theta$.

(11) $\dfrac{1-\cos\alpha}{\sin\alpha} = \tan\dfrac{\alpha}{2}$.

(12) $\sin 2\phi = \dfrac{2\sqrt{(\operatorname{cosec}^2\phi - 1)}}{\operatorname{cosec}^2\phi}$.

(13) $\cos 2\phi = \dfrac{2-\sec^2\phi}{\sec^2\phi}$.

(14) $\tan 2\phi = \dfrac{2\cot\phi}{\cot^2\phi - 1}$.

(15) $\sin\alpha = \sqrt{\left(\dfrac{\sec 2\alpha - 1}{2\sec\alpha}\right)}$.

(16) $\cos\alpha = \sqrt{\left(\dfrac{\sec 2\alpha + 1}{2\sec\alpha}\right)}$.

(17) $\tan\alpha = \operatorname{cosec} 2\alpha - \cot 2\alpha.$

(18) $\cot\beta = \operatorname{cosec} 2\beta + \cot 2\beta.$

(19) $\tan(45^\circ + A) = \dfrac{\cos 2A}{1 - \sin 2A}.$

(20) $\cot(45^\circ - A) = \sec 2A + \tan 2A.$

(21) $\dfrac{1 + \sin\alpha}{1 + \cos\alpha} = \dfrac{1}{2}\left(1 + \tan\dfrac{\alpha}{2}\right)^2.$

(22) $\dfrac{1 - \sin\alpha}{1 - \cos\alpha} = \dfrac{1}{2}\left(\cot\dfrac{\alpha}{2} - 1\right)^2.$

(23) $\tan\theta = \tan\dfrac{\theta}{2} + \dfrac{1}{2}\tan\theta \,.\, \sec^2\dfrac{\theta}{2}.$

(24) $\dfrac{1 + \sin\theta}{1 - \sin\theta} = (\sec\theta + \tan\theta)^2.$

(25) $\sin 2A = \dfrac{1 - \tan^2(45^\circ - A)}{1 + \tan^2(45^\circ + A)}.$

(26) $\sin 2\theta = \dfrac{\tan\left(\dfrac{\pi}{4} + \theta\right) - \tan\left(\dfrac{\pi}{4} - \theta\right)}{\tan\left(\dfrac{\pi}{4} + \theta\right) + \tan\left(\dfrac{\pi}{4} - \theta\right)}.$

126. $\sin 3A = \sin(2A + A)$

$$
\begin{aligned}
&= \sin 2A \,.\, \cos A + \cos 2A \,.\, \sin A \\
&= (2\sin A \,.\, \cos A) \,.\, \cos A + (1 - 2\sin^2 A) \,.\, \sin A \\
&= 2\sin A \,.\, \cos^2 A + \sin A - 2\sin^3 A \\
&= 2\sin A \,.\, (1 - \sin^2 A) + \sin A - 2\sin^3 A \\
&= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A \\
&= 3\sin A - 4\sin^3 A.
\end{aligned}
$$

127.
$$\begin{aligned}\cos 3A &= \cos(2A + A)\\ &= \cos 2A\,.\,\cos A - \sin 2A\,.\,\sin A\\ &= (2\cos^2 A - 1)\,.\,\cos A - (2\sin A\,.\,\cos A)\,.\,\sin A\\ &= 2\cos^3 A - \cos A - 2\sin^2 A\,.\,\cos A\\ &= 2\cos^3 A - \cos A - 2\cos A\,(1 - \cos^2 A)\\ &= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A\\ &= 4\cos^3 A - 3\cos A.\end{aligned}$$

128.
$$\begin{aligned}\text{Tan } 3A &= \frac{\sin 3A}{\cos 3A}\\ &= \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}, \text{ and, dividing by } \cos^3 A\\ &= \frac{\dfrac{3\sin A}{\cos^3 A} - 4\tan^3 A}{4 - \dfrac{3}{\cos^2 A}}\\ &= \frac{3\tan A\,.\,\sec^2 A - 4\tan^3 A}{4 - 3\sec^2 A}\\ &= \frac{3\tan A\,(1 + \tan^2 A) - 4\tan^3 A}{4 - 3\,(1 + \tan^2 A)}\\ &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.\end{aligned}$$

Examples.—XXXV.

In some of these Examples the student may employ with advantage the formulæ given in Art. 116.

1. (1) $\dfrac{\cos 3\theta - \sin 3\theta}{\sin\theta + \cos\theta} = 1 - 2\sin 2\theta.$

(2) $\sin 2\theta + \cos\theta = \dfrac{2\tan\theta + \sec\theta}{1 + \tan^2\theta}.$

(3) $\sin A = \tan\frac{A}{2} + 2\sin^2\frac{A}{2}\cot A.$

(4) $\dfrac{\cot A}{\cot A - \cot 3A} + \dfrac{\tan A}{\tan A - \tan 3A} = 1.$

(5) $\cos 4A + \cos 4B = 2\{1 - 2\sin^2(A+B)\}\{1 - 2\sin^2(A-B)\}.$

(6) $\tan(45^\circ + \theta) - \tan(45^\circ - \theta) = 2 . \dfrac{\sec 2\theta - \cos 2\theta}{\sin 2\theta}.$

(7) $\cot^2\theta - \tan^2\theta = 8\dfrac{\cos 2\theta}{1 - \cos 4\theta}.$

(8) $2\sin A . \cos 2A = \sin 3A - \sin A.$

(9) $\dfrac{\cos nA - \cos(n+2)A}{\sin(n+2)A - \sin nA} = \tan(n+1)A.$

(10) $\cos 9A + 3\cos 7A + 3\cos 5A + \cos 3A = 8\cos^3 A . \cos 6A.$

(11) $\dfrac{\operatorname{cosec} 2A - \cot 2A}{\operatorname{cosec} 2A + \cot 2A} = \tan^2 A.$

(12) $\dfrac{1 + \sin A}{1 + \cos A} = \dfrac{1}{2}\left(1 + \tan\dfrac{A}{2}\right)^2.$

(13) $\dfrac{2\cos 2A - 3}{2\cos 2A + 3} = \dfrac{\cos 3A - 2\cos A}{\sin 3A + 2\sin A} . \tan A.$

(14) $\tan(45^\circ - A) + \tan(45^\circ + A) = 2\sec 2A.$

(15) $\cos\alpha - \tan\frac{\alpha}{2}\sin\alpha = \cos 2\alpha + \tan\frac{\alpha}{2} . \sin 2\alpha.$

(16) $\cot^2 A - \tan^2 A = 4\cot 2A . \operatorname{cosec} 2A.$

(17) $\operatorname{cosec}\alpha . \cot\alpha - \sec\alpha . \tan\alpha = 4\operatorname{cosec} 2\alpha . \cot 2\alpha.$

(18) $\cot^2\alpha - \tan^2\alpha = \dfrac{4\cos 2\alpha}{\sin^2 2\alpha}.$

(19) $\operatorname{cosec}^2 b - \sec^2 b = 4 \cos 2b \,.\, \operatorname{cosec}^2 2b.$

(20) $\cot^2\left(45^\circ + \frac{A}{2}\right) = \frac{2 \operatorname{cosec} 2A - \sec A}{2 \operatorname{cosec} 2A + \sec A}.$

(21) $\sin\left(\frac{5\pi}{2} + \theta\right) - \sin\left(\frac{3\pi}{2} - \theta\right) = \sin\left(\frac{5\pi}{2} - \theta\right) - \sin\left(\frac{3\pi}{2} + \theta\right).$

22. $\cot\left(\frac{\pi}{2} + \theta\right) - \tan\left(\frac{\pi}{2} + \theta\right) = 2 \cot 2\theta.$

(23) $\frac{(\operatorname{cosec} \alpha + \sec \alpha)^2}{\operatorname{cosec}^2 \alpha + \sec^2 \alpha} = 1 + \sin 2\alpha.$

(24) $\frac{\tan \theta}{\tan 2\theta - \tan \theta} = \cos 2\theta.$ (25) $\frac{\tan 2\theta \,.\, \tan \theta}{\tan 2\theta - \tan \theta} = \sin 2\theta.$

(26) $\tan(\alpha + \beta) = \frac{\sin^2 \alpha - \sin^2 \beta}{\sin \alpha \,.\, \cos \alpha - \sin \beta \,.\, \cos \beta}.$

(27) $4 \sin A \,.\, \sin(60^\circ + A) \,.\, \sin(60^\circ - A) = \sin 3A.$

(28) $\operatorname{cosec} 2\theta + \cot 4\theta + \operatorname{cosec} 4\theta = \cot \theta.$

2. Solve the equations :

(1) $\sin 2\theta + \sqrt{3} \,.\, \cos 2\theta = 1.$ (2) $\sin^2 2\theta - \sin^2 \theta = \sin^2 \frac{\pi}{4}.$

(3) $\sin 5x . \cos 3x = \sin 9x . \cos 7x.$ (4) $2 \sin^2 3\theta + \sin^2 6\theta = 2.$

(5) $\cos 2A + \sin^2 A = \frac{3}{4}.$ (6) $\sin 2\theta = \sin^2 3\theta.$

(7) $4 \sin \theta \,.\, \sin 3\theta = 1.$ (8) $\tan 2\alpha = 3 \tan \alpha.$

(9) $2 \cos 2\theta = 2 \sin \theta + 1.$ (10) $\sin 7\alpha - \sin \alpha = \sin 3\alpha.$

(11) $\operatorname{cosec}^2 \theta - \sec^2 \theta = 2 \operatorname{cosec}^2 \theta \,.\, \sqrt{3}.$

(12) $\sin 6\theta = 2 \sin 4\theta - \sin 2\theta.$

129. To find the Trigonometrical Ratios for an angle of 18°.

Let $A = 18°.$

Then $2A = 36°,$

and $3A = 54°.$

Now the sine of an angle is equal to the cosine of the complement of the angle;

$$\therefore \sin 36° = \cos 54°;$$

$$\therefore \sin 2A = \cos 3A;$$

$$\therefore 2\sin A \,.\, \cos A = 4\cos^3 A - 3\cos A.$$

Divide by $\cos A$.

Then
$$2\sin A = 4\cos^2 A - 3,$$

$$2\sin A = 4(1 - \sin^2 A) - 3,$$

$$2\sin A = 4 - 4\sin^2 A - 3,$$

$$4\sin^2 A + 2\sin A = 1,$$

$$\sin^2 A + \frac{1}{2} \,.\, \sin A = \frac{1}{4},$$

$$\sin^2 A + \frac{1}{2}\sin A + \frac{1}{16} = \frac{5}{16},$$

$$\sin A + \frac{1}{4} = \frac{\pm\sqrt{5}}{4},$$

and taking the upper sign, since sin 18° must be positive,

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

Hence we can find cos 18°, tan 18° and the other ratios.

Examples.—XXXVI.

Given $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$ find the value of the following Ratios.

(1) sin 36°. (2) cos 36°. (3) sin 54°. (4) cos 54°.

(5) sin 72°. (6) tan 72°. (7) sin 90°. (8) cos 90°.

Shew that

$$\sin(36^\circ + A) + \sin(72^\circ - A) - \sin(36^\circ - A) - \sin(72^\circ + A) = \sin A,$$

and that

$$\sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A) = \cos A.$$

130. We now proceed to the formulæ relating to submultiples of angles, and first we shall prove that

$$\sin\frac{A}{2} = \pm\sqrt{\left(\frac{1-\cos A}{2}\right)}.$$

Since $\cos A = 1 - 2\sin^2\frac{A}{2}$; (Art. 124,)

$$2\sin^2\frac{A}{2} = 1 - \cos A;$$

$$\therefore \sin^2\frac{A}{2} = \frac{1-\cos A}{2};$$

$$\therefore \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}.$$

If the value of A be given, we know whether $\sin\frac{A}{2}$ is positive or negative, and hence we know which sign is to be taken. Thus, if A be between 0° and 360°, $\sin\frac{A}{2}$ lies between 0° and 180°, and is positive: but if A be between 360° and 720°, $\sin\frac{A}{2}$ lies between 180° and 360° and is negative.

131. We shall next shew that

$$\cos\frac{A}{2} = \pm\sqrt{\left(\frac{1+\cos A}{2}\right)}.$$

Since $\cos A = 2\cos^2\frac{A}{2} - 1$; (Art. 124,)

$$\therefore -2\cos^2\frac{A}{2} = -1 - \cos A;$$

$$\therefore\ 2\cos^2\frac{A}{2} = 1 + \cos A;$$

$$\therefore\ \cos^2\frac{A}{2} = \frac{1+\cos A}{2};$$

$$\therefore\ \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}.$$

If the value of A be given, we know whether $\cos\frac{A}{2}$ is positive or negative, and thus we know which sign is to be taken. For instance, if A lies between 0° and 180°, $\cos\frac{A}{2}$ is positive: but if A lies between 180° and 360°, $\cos\frac{A}{2}$ is negative.

132. To prove that

$$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

and $$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}.$$

Since $$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1,$$

and $$2\sin\frac{A}{2}\,.\,\cos\frac{A}{2} = \sin A;$$

$$\therefore\ \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\,.\,\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 + \sin A,$$

and $$\sin^2\frac{A}{2} - 2\sin\frac{A}{2}\,.\,\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 - \sin A.$$

Hence, taking the square root of each side of both equations,

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A},$$

and
$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}.$$

Therefore, by addition,

$$2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A},$$

and, by subtraction,

$$2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}.$$

133. If the value of A be given we know the *signs* of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, as we explained in the preceding articles. We also know whether $\sin\frac{A}{2}$ is greater or less than $\cos\frac{A}{2}$. Hence, if A be known, we can assign with certainty the signs which the root-symbols are to have in the *intermediate* equations

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A},$$

and hence we can select the proper signs in the *final* equations.

For instance, suppose A to lie between 180° and 270°.

Then $\frac{A}{2}$ lies between 90° and 135°.

Therefore $\sin\frac{A}{2}$ is positive and $\cos\frac{A}{2}$ is negative.

Also, for an angle between 90° and 135° the sine is numerically greater than the cosine.

Hence we must take the positive sign in both equations.

EXAMPLES.—XXXVII.

1. Affix to the root-symbols the proper signs when A is 15°.

2. Affix to the root-symbols the proper signs when A is 300°.

3. If $\sin 378^\circ = \frac{\sqrt{5}-1}{4}$, determine $\cos 189^\circ$ and $\sin 189^\circ$.

4. If $\sin 19^\circ.\, 29' = \frac{1}{3}$, what is the value of $\sin 9^\circ.\, 44'.\, 30''$?

5. If $\cos 315^\circ = \frac{1}{\sqrt{2}}$, find the value of $\cos 157^\circ.\, 30'$.

134. We mentioned in Art. 109 what are called the Inverse Trigonometrical Functions, $\sin^{-1} x$, $\cos^{-1} x$, &c. We shall now give an example to illustrate the method of combining these functions.

To prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$.

Let $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$;

then $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$.

Now
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \,.\, \tan\beta}$$
$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$
$$= \frac{5}{5}$$
$$= 1;$$
$$\therefore\ \alpha + \beta = 45^\circ,$$

that is, $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$.

Examples.—XXXVIII.

(1) If $A = \sin^{-1}\frac{3}{5}$ and $B = \sin^{-1}\frac{4}{5}$, shew that $A + B = 90°$.

(2) If $A = \tan^{-1}\frac{1}{7}$ and $B = \tan^{-1}\frac{1}{3}$, shew that $A + 2B = 45°$.

(3) Shew that $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3 = 45°$.

(4) Shew that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45°$.

(5) Shew that $\cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7} = 135°$.

(6) Shew that $\tan^{-1}\frac{3}{5} + \cot^{-1}\frac{7}{3} = \cot^{-1}\frac{13}{18}$.

(7) Shew that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x - y}{1 + xy}$.

(8) Shew that $\sin^{-1} x + \cos^{-1} x = 90°$.

(9) Shew that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$.

(10) Shew that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$.

CHAPTER XII.

ON LOGARITHMS.

135. **Logarithms** are numbers arranged in tables for the purpose of facilitating arithmetical computations.

They are adapted to the natural numbers 1, 2, 3...in such a manner that by means of them

the operation of Multiplication is changed into that of Addition,

.... Division.................................... Subtraction,

.... Involution............................... Multiplication,

.... Evolution .... Division.

136. Def. The Logarithm of a number to a given base is the index of the power to which the base must be raised to give the number.

Thus if $m = a^x$, x is called the logarithm of m to the base a.

For instance, if the base of a system of Logarithms be 2,

3 is the logarithm of the number 8,

because $8 = 2^3$:

and if the base be 5, then

3 is the logarithm of the number 125,

because $125 = 5^3$.

137. The logarithm of a number m to the base a is written thus, $\log_a m$; and so, if $m = a^x$,

$$x = \log_a m.$$

Hence it follows that $m = a^{\log_a m}$.

138. Since $1 = a^0$, the logarithm of unity to any base is zero.
Since $a = a^1$, the logarithm of the base of any system is unity.

139. We now proceed to describe that which is called the Common System of logarithms.

The base of the system is 10.

By a *system* of logarithms to the base 10, we mean a succession of values of x which satisfy the equation

$$m = 10^x$$

for all positive values of m, integral or fractional.

Such a system is formed by the series of logarithms of the natural numbers from 1 to 100000, which constitute the logarithms registered in our ordinary tables, and which are therefore called *tabular logarithms*.

140. Now

$$\begin{aligned} 1 &= 10^0, \\ 10 &= 10^1, \\ 100 &= 10^2, \\ 1000 &= 10^3, \end{aligned}$$

and so on.

Hence the logarithm of 1 is 0,
of 10 is 1,
of 100 is 2,
of 1000 is 3,

and so on.

Hence for all numbers between 1 and 10 the logarithm is a decimal less than 1,

between 10 and 100 the logarithm is a decimal between 1 and 2,
between 100 and 1000 .... a decimal between 2 and 3,

and so on.

141. The logarithms of the natural numbers from 1 to 12 stand thus in the tables:

No.	Log	No.	Log
1	0·0000000	7	0·8450980
2	0·3010300	8	0·9030900
3	0·4771213	9	0·9542425
4	0·6020600	10	1·0000000
5	0·6989700	11	1·0413927
6	0·7781513	12	1·0791812

The logarithms are calculated to seven places of decimals.

142. The integral parts of the logarithms of numbers higher than 10 are called the *characteristics* of those logarithms, and the decimal parts of the logarithms are called the *mantissæ*.

Thus 1 is the characteristic,
·0791812 the mantissa,
of the logarithm of 12.

143. The logarithms for 100 and the numbers that succeed it (and in some tables those that precede 100) have no characteristic prefixed, because it can be supplied by the reader, being 2 for all numbers between 100 and 1000, 3 for all between 1000 and 10000, and so on. Thus in the Tables we shall find

No.	Log
100	0000000
101	0043214
102	0086002

which we read thus:

the logarithm of 100 is 2,
of 101 is 2·0043214,
of 102 is 2·0086002.

144. We shall now prove the properties of logarithms, of which the fundamental one is the following:

The logarithm of a product is equal to the sum of the logarithms of its factors.

Let $m = a^x$,

and $n = a^y$.

Then $mn = a^{x+y}$;

$$\therefore \log_a mn = x + y$$
$$= \log_a m + \log_a n.$$

Hence it follows that

$$\log_a mnp = \log_a m + \log_a n + \log_a p,$$

and similarly it may be shewn that the Theorem holds good for any number of factors.

Thus the operation of Multiplication is changed into that of Addition.

Suppose, for instance, we want to find the product of 246 and 357, we add the logarithms of the factors, and the sum is the logarithm of the product: thus

$$\log 246 = 2\cdot 3909351$$
$$\log 357 = 2\cdot 5526682$$
$$\text{their sum} = 4\cdot 9436033$$

which is the logarithm of 87822, the product required.

NOTE. We do not write $\log_{10} 246$, for so long as we are treating of logarithms to the particular base 10, we may omit the suffix.

145. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

Let $m = a^x$,

and $n = a^y$.

Then $\frac{m}{n} = a^{x-y}$;

$$\therefore \log_a \frac{m}{n} = x - y$$
$$= \log_a m - \log_a n.$$

Thus the operation of Division is changed into that of Subtraction.

If, for example, we are required to divide 371·49 by 52·376, we proceed thus

$$\begin{aligned} \log 371{\cdot}49 &= 2{\cdot}5699471 \\ \log 52{\cdot}376 &= 1{\cdot}7191323 \\ \text{their difference} &= {\cdot}8508148 \end{aligned}$$

which is the logarithm of 7·092752, the quotient required.

146. *The logarithm of any power of a number is equal to the product of the logarithm of the number and the index denoting the power.*

Let $m = a^x$.

Then $m^r = a^{rx}$;

$$\therefore \log m^r = rx$$
$$= r \, . \log_a m.$$

Thus the operation of Involution is changed into Multiplication.

Suppose, for instance, we have to find the fourth power of 13, we may proceed thus

$$\begin{array}{r} \log 13 = 1{\cdot}1139434 \\ 4 \\ \hline 4{\cdot}4557736 \end{array}$$

which is the logarithm of 28561, the number required.

147. *The logarithm of any root of a number is equal to the quotient arising from the division of the logarithm of the number by the number denoting the root.*

Let $m = a^x$.

Then $m^{\frac{1}{r}} = a^{\frac{x}{r}}$;

$$\therefore \log_a m^{\frac{1}{r}} = \frac{x}{r}$$
$$= \frac{1}{r} \, . \log_a m.$$

Thus the operation of Evolution is changed into Division.

If, for example, we have to find the fifth root of 16807, we proceed thus

$$\begin{array}{r|l} 5 & 4{\cdot}2254902, \text{ the log of } 16807 \\ \hline & {\cdot}8450980 \end{array}$$

which is the logarithm of 7, the root required.

148. The common system of Logarithms has this advantage over all others for numerical calculations that its base is the same as the radix of the common scale of notation.

Hence it is that the same mantissa serves for all numbers which have the same significant digits and differ only in the position of the place of units relatively to those digits.

For, since

$$\begin{aligned} \log 60 &= \log 10 + \log 6 = 1 + \log 6, \\ \log 600 &= \log 100 + \log 6 = 2 + \log 6, \\ \log 6000 &= \log 1000 + \log 6 = 3 + \log 6, \end{aligned}$$

it is clear that if we know the logarithm of any number, as 6, we also know the logarithms of the numbers resulting from multiplying that number by the powers of 10.

So again, if we know that

$$\log 1{\cdot}7692 \text{ is } {\cdot}247783,$$

we also know that

$$\begin{aligned} &\log 17{\cdot}692 \text{ is } 1{\cdot}247783, \\ &\log 176{\cdot}92 \text{ is } 2{\cdot}247783, \\ &\log 1769{\cdot}2 \text{ is } 3{\cdot}247783, \\ &\log 17692 \text{ is } 4{\cdot}247783, \\ &\log 176920 \text{ is } 5{\cdot}247783. \end{aligned}$$

149. We must now treat of the logarithms of numbers less than unity.

Since

$$\begin{aligned} 1 &= 10^0, \\ {\cdot}1 &= \frac{1}{10} = 10^{-1}, \\ {\cdot}01 &= \frac{1}{100} = 10^{-2}, \end{aligned}$$

the logarithm of a number

.... between 1 and ·1 lies between 0 and − 1,
.... between ·1 and ·01 .... − 1 and − 2,
.... between ·01 and ·001 .... − 2 and − 3,

and so on.

Hence the logarithms of all numbers less than unity are Negative.

We do not require a separate table for these logarithms, for we can deduce them from the logarithms of numbers greater than unity by the following process:

$$\log \cdot 6 = \log \frac{6}{10} = \log 6 - \log 10 = \log 6 - 1,$$

$$\log \cdot 06 = \log \frac{6}{100} = \log 6 - \log 100 = \log 6 - 2,$$

$$\log \cdot 006 = \log \frac{6}{1000} = \log 6 - \log 1000 = \log 6 - 3.$$

Now the logarithm of 6 is ·7781513.

Hence

$\log \cdot 6 = -1 + \cdot 7781513$, which is written $\bar{1}\cdot 7781513$,
$\log \cdot 06 = -2 + \cdot 7781513$, which is written $\bar{2}\cdot 7781513$,
$\log \cdot 006 = -3 + \cdot 7781513$, which is written $\bar{3}\cdot 7781513$,

the characteristics only being negative and the mantissæ positive.

150. Thus the same mantissæ serve for the Logarithms of all numbers, *whether greater or less than unity*, which have the same significant digits and differ only in the position of the place of units relatively to those digits.

It is best to regard the Table as a register of the logarithms of numbers which have *one* significant digit before the decimal point.

For instance, when we read in the tables

No.	Log
144	1583625

to interpret the entry thus

log 1·44 is ·1583625.

We then obtain the following rules for the characteristic to be attached in each case.

I. If the decimal point be shifted one, two, three....n places to the right, prefix as a characteristic 1, 2, 3....n.

II. If the decimal point be shifted one, two, three....n places to the left, prefix as a characteristic $\bar{1}$, $\bar{2}$, $\bar{3}$....$\bar{n}$.

Thus log 1·44 is ·1583625,

∴ log 14·4 is 1·1583625,

log 144 is 2·1583625,

log 1440 is 3·1583625,

and log ·144 is $\bar{1}$·1583625,

log ·0144 is $\bar{2}$·1583625,

log ·00144 is $\bar{3}$·1583625.

151. In calculations with negative characteristics we follow the rules of algebra. Thus

(1) If we have to add the logarithms $\bar{3}$·64628 and 2·42367, we first add the mantissæ, and the result is 1·06995, and then add the characteristics, and this result is $\bar{1}$.

The final result is $\bar{1}+1{\cdot}06995$, that is, ·06995.

(2) To subtract $\bar{5}$·6249372 from $\bar{3}$·2456973, we may arrange the numbers thus

$$\begin{array}{r} -3+{\cdot}2456973 \\ -5+{\cdot}6249372 \\ \hline 1+{\cdot}6207601 \end{array}$$

the 1 carried on from the last subtraction in the decimal places changing -5 into -4, and then -4 subtracted from -3 giving 1 as a result.

Hence the resulting logarithm is 1·6207601.

(3) To multiply $\bar{3}\cdot7482569$ by 5.

$$\begin{array}{r} \bar{3}\cdot7482569 \\ 5 \\ \hline \overline{12}\cdot7412845 \end{array}$$

the 3 carried on from the last multiplication of the decimal places being added to -15, and thus giving -12 as a result.

(4) To divide $\overline{14}\cdot2456736$ by 4.

Increase the negative characteristic so that it may be exactly divisible by 4, making a proper compensation, thus

$$\overline{14}\cdot2456736 = \overline{16} + 2\cdot2456736.$$

Then $$\frac{\overline{14}\cdot2456736}{4} = \frac{\overline{16} + 2\cdot2456736}{4} = \bar{4} + \cdot5614184;$$

and so the result is $\bar{4}\cdot5614184$.

EXAMPLES.—XXXIX.

(1) Add $\bar{3}\cdot1651553$, $\bar{4}\cdot7505855$, $6\cdot6879746$, $\bar{2}\cdot6150026$.

(2) Add $\bar{4}\cdot6843785$, $\bar{5}\cdot6650657$, $3\cdot8905196$, $\bar{3}\cdot4675284$.

(3) Add $2\cdot5324716$, $3\cdot6650657$, $\bar{5}\cdot8905196$, $\cdot3156215$.

(4) From $2\cdot483269$ take $\bar{3}\cdot742891$.

(5) From $\bar{2}\cdot352678$ take $\bar{5}\cdot428619$.

(6) From $\bar{5}\cdot349162$ take $\bar{3}\cdot624329$.

(7) Multiply $\bar{2}\cdot4596721$ by 3.

(8) Multiply $\bar{7}\cdot429683$ by 6.

(9) Multiply $\bar{9}\cdot2843617$ by 7.

(10) Divide $\bar{6}\cdot3725409$ by 3.

(11) Divide $\overline{14}\cdot432962$ by 6.

(12) Divide $\bar{4}\cdot53627188$ by 9.

152. We shall now explain how a system of logarithms calculated to a base a may be transformed into another system of which the base is b.

Let m be a number of which the logarithm in the first system is x and in the second y.

Then $$m = a^x,$$
and $$m = b^y.$$
Hence $$b^y = a^x,$$
$$\therefore b = a^{\frac{x}{y}};$$
$$\therefore \frac{x}{y} = \log_a b;$$
$$\therefore \frac{y}{x} = \frac{1}{\log_a b};$$
$$\therefore y = \frac{1}{\log_a b}x.$$

Hence if we multiply the logarithm of any number in the system of which the base is a by $\frac{1}{\log_a b}$, we shall obtain the logarithm of the same number in the system of which the base is b.

This constant multiplier $\frac{1}{\log_a b}$ is called THE MODULUS *of the system of which the base is b* with reference to the system of which the base is a.

153. The common system of logarithms is used in all numerical calculations, but there is another system, which we must notice, employed by the discoverer of logarithms, Baron Napier, and hence called The Napierian System.

The base of this system, denoted by the symbol e, is the number which is the sum of the series

$$2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \dots \textit{ad inf.},$$

of which sum the first eight digits are 2·7182818.

154. Our common logarithms are formed from the Logarithms of the Napierian System by multiplying each of the latter by a common multiplier called The Modulus of the Common System.

This modulus is, in accordance with the conclusion of Art. 152, $\frac{1}{\log_e 10}$.

That is, if l and N be the logarithms of the same number in the common and Napierian systems respectively,

$$l = \frac{1}{\log_e 10} \cdot N.$$

Now $\log_e 10$ is 2·30258509;

$$\therefore \frac{1}{\log_e 10} \text{ is } \frac{1}{2{\cdot}30258509} \text{ or } {\cdot}43429448,$$

and so the modulus of the common system is ·43429448.

155. To prove that $\log_a b \times \log_b a = 1$.

Let x be the logarithm of m to the base a,

y .... to the base b.

Then $$m = a^x,$$

and $$m = b^y;$$

$$\therefore a^x = b^y.$$

Hence $$a = b^{\frac{y}{x}},$$

and $$b = a^{\frac{x}{y}};$$

$$\therefore \frac{y}{x} = \log_b a,$$

and $$\frac{x}{y} = \log_a b;$$

$$\therefore \log_a b \times \log_b a = \frac{y}{x} \times \frac{x}{y}$$
$$= 1.$$

156. The following are simple examples of the method of applying the principles explained in this Chapter.

Ex. (1). Given $\log 2 = \cdot 3010300$ and $\log 3 = \cdot 4771213$, find the logarithms of 64, 81 and 96.

$$\log 64 = \log 2^6 = 6 \log 2$$

$$\begin{array}{rr} \log 2 = & \cdot 3010300 \\ & 6 \\ \hline \therefore \log 64 = & 1\cdot 8061800 \\ \hline \end{array}$$

$$\log 81 = \log 3^4 = 4 \log 3$$

$$\begin{array}{rr} \log 3 = & \cdot 4771213 \\ & 4 \\ \hline \therefore \log 81 = & 1\cdot 9084852 \\ \hline \end{array}$$

$$\log 96 = \log (32 \times 3) = \log 32 + \log 3,$$

and $$\log 32 = \log 2^5 = 5 \log 2;$$

$$\therefore \log 96 = 5 \log 2 + \log 3 = 1\cdot 5051500 + \cdot 4771213 = 1\cdot 9822713.$$

Ex. (2). Given $\log 5 = \cdot 6989700$, find the logarithm of $\sqrt[7]{(6\cdot 25)}$.

$$\log (6\cdot 25)^{\frac{1}{7}} = \frac{1}{7} \log 6\cdot 25 = \frac{1}{7} \log \frac{625}{100} = \frac{1}{7} (\log 625 - \log 100)$$

$$= \frac{1}{7} (\log 5^4 - 2) = \frac{1}{7} (4 \log 5 - 2)$$

$$= \frac{1}{7} (2\cdot 7958800 - 2) = \cdot 1136657.$$

EXAMPLES.—XL.

(1) Given $\log 2 = \cdot 3010300$, find $\log 128$, $\log 125$ and $\log 2500$.

(2) Given $\log 2 = \cdot 3010300$ and $\log 7 = \cdot 8450980$, find the logarithms of 50, $\cdot 005$ and 196.

(3) Given log 2 = ·3010300, and log 3 = ·4771213, find the logarithms of 6, 27, 54 and 576.

(4) Given log 2 = ·3010300, log 3 = ·4771213, log 7 = ·8450980, find log 60, log ·03, log 1·05, and log ·0000432.

(5) Given log 2 = ·3010300, log 18 = 1·2552725 and log 21 = 1·3222193, find log ·00075 and log 31·5.

(6) Given log 5 = ·6989700, find the logarithms of 2, ·064, and $\left(\frac{2^{60}}{5^{20}}\right)^{\frac{1}{14}}$.

(7) Given log 2 = ·3010300, find the logarithms of 5, ·125 and $\left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}}$.

(8) What are the logarithms of ·01, 1 and 100 to the base 10? What to the base ·01?

(9) What is the characteristic of log 1593, (1) to base 10, (2) to base 12?

(10) Given $\frac{4^x}{2^{x+y}} = 8$, and $x = 3y$, find x and y.

(11) Given log 4 = ·6020600, log 1·04 = ·0170333:

(*a*) Find the logarithms of 2, 25, 83·2, $(\cdot 625)^{\frac{1}{100}}$.

(*b*) How many digits are there in the integral part of $(1\cdot 04)^{6000}$?

(*c*) In how many years will a sum of money double itself, at 4 per cent. compound interest, payable yearly?

(12) Given log 25 = 1·3979400, log 1·03 = ·0128372:

(*a*) Find the logarithms of 5, 4, 51·5, $(\cdot 064)^{\frac{1}{100}}$.

(*b*) How many digits are there in the integral part of $(1\cdot 03)^{600}$?

(*c*) In how many years will a sum of money double itself at 3 per cent. compound interest, payable yearly?

(13) Having given $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$,

$$\log 11 = 1\cdot 0413927:$$

find the logarithms of 7623, $\frac{77}{300}$ and $\frac{3}{539}$.

(14) Solve the equations:

(1) $4\cdot 096^x = \frac{8}{64^x}$. (4) $a^{mx} b^{2x} = c$.

(2) $\left(\frac{1}{\cdot 4}\right)^x = 6\cdot 25$. (5) $a^{3x} . b^{4-x} = c^{2x-1}$.

(3) $a^x . b^x = m$. (6) $a^x b^{mx} = c^{1-3x}$.

CHAPTER XIII.

ON TRIGONOMETRICAL AND LOGARITHMIC TABLES.

157. We shall give in this Chapter a short description of the Tables which have been constructed for the purpose of facilitating Trigonometrical calculations.

The methods by which these Tables are formed do not fall within the range of this treatise : we have merely to explain how they are applied to the solution of such simple examples as we shall hereafter give.

We shall arrange our remarks in the following order :

I. On Tables of Logarithms of Numbers.

II. On Tables of Trigonometrical Ratios.

III. On Tables of Logarithms of Trigonometrical Ratios.

I. *On Tables of Logarithms of Numbers.*

158. These Tables are arranged so as to give the mantissæ of the logarithms of the natural numbers from 1 to 10000, that is of numbers containing from one to five digits.

We shall now shew how by aid of these tables, first, to find the logarithm of any given number, and, secondly, how to determine the number which corresponds to a given logarithm.

159. *When a number is given to find its logarithm.*

When the given number has not more than five digits we can take its logarithm at once from the tables.

When the given number has more than five digits we can determine its logarithm by a process, which will be best explained by an example.

Suppose we require to find the logarithm of 6276153.

We find from the tables

log 62761 is 4·7976899,

and log 62762 is 4·7976988.

Hence log 6276200 is 6·7976988,

and log 6276100 is 6·7976899.

Thus for a difference of 100 in the numbers the difference of the logarithms is ·0000089.

We then reason thus: If we have to add ·0000089 to the logarithm of 6276100 to obtain the logarithm of 6276200, what must we add to the logarithm of 6276100 to obtain the logarithm of 6276153?

Assuming that the increase of the logarithm is proportional to the increase of the number (which is nearly but not quite true), we shall have

$$100 : 53 :: \cdot 0000089 : \text{that which we have to add};$$

$$\therefore \text{ number to be added} = \frac{53 \times \cdot 0000089}{100} = \cdot 000004717,$$

and therefore, omitting the last two figures,

$$\log 6276153 = 6 \cdot 7976899 + \cdot 0000047$$
$$= 6 \cdot 7976946.$$

If the first of the figures omitted be 5 or a digit higher than 5, it is usual to increase the figure immediately preceding it by 1: thus if the number to be added had been ·000004757 we should have added ·0000048.

160. We took in the last article an *integral* number, but the same process will apply to numbers containing decimals.

For suppose we have to find the logarithm of 627·6153: we can find log 6276153 as before, and the only difference to be made in the final result is to change the characteristic from 6 to 2, that is,

$$\log 627{\cdot}6153 = 2{\cdot}7976946.$$

So again, in accordance with Art. 149,

$$\log {\cdot}00006276153 = \bar{5}{\cdot}7976946.$$

EXAMPLES.—XLI.

(1) Given $\log 52502 = 4{\cdot}7201758,$
$\log 52503 = 4{\cdot}7201841,$
find $\log 52502{\cdot}5.$

(2) Given $\log 3{\cdot}0042 = {\cdot}4777288,$
$\log 3{\cdot}0043 = {\cdot}4777433,$
find $\log 300{\cdot}425.$

(3) Given $\log 3202{\cdot}5 = 3{\cdot}5054891,$
$\log 3202{\cdot}6 = 3{\cdot}5055027,$
find $\log 32{\cdot}025613.$

(4) Given $\log 23660 = 4{\cdot}3740147,$
$\log 23661 = 4{\cdot}3740331,$
find $\log 236{\cdot}601.$

(5) Given $\log 67502 = 4{\cdot}8293166,$
$\log 67503 = 4{\cdot}8293231,$
find $\log 67{\cdot}5021.$

(6) Given $\log 73335 = 4{\cdot}8653113,$
$\log 73336 = 4{\cdot}8653172,$
find $\log {\cdot}007333533.$

(7) Given $\log 65931 = 4{\cdot}8190897$,

$\log 65932 = 4{\cdot}8190962$,

find $\log {\cdot}000006593171$.

(8) Given $\log 34{\cdot}077 = 1{\cdot}5324614$,

$\log 34{\cdot}078 = 1{\cdot}5324741$,

find $\log 3407{\cdot}78$.

(9) Given $\log 39097 = 4{\cdot}5921434$,

$\log 39098 = 4{\cdot}5921545$,

find $\log 390974$.

(10) Given $\log 25819 = 4{\cdot}4119394$,

$\log 25820 = 4{\cdot}4119562$,

find $\log 2{\cdot}581926$.

161. *When a logarithm is given, to find the number to which it corresponds.*

If the decimal part of the logarithm is found exactly in the tables, we can take out the corresponding number.

Thus if we have to find the number corresponding to the logarithm 2·8598645, we look in the tables for the mantissa ·8598645, and we find it set down opposite the number 72421, hence

2·8598645 is the logarithm of 724·21.

Next, suppose that the decimal part of the logarithm is not found exactly in the tables, and that we have to find the number corresponding to the logarithm 3·9212074.

We find from the tables

$$\log 8340{\cdot}8 = 3{\cdot}9212077,$$

and

$$\log 8340{\cdot}7 = 3{\cdot}9212025.$$

Hence a difference of ·1 in the numbers gives a difference of ·0000052 in the logarithms.

Then we reason thus: If we must add ·1 to 8340·7 for an excess of ·0000052 above the logarithm of 8340·7, what must we add for an excess of (3·9212074 − 3·9212025) or ·0000049?

Assuming that the increase of the number is proportional to the increase of the logarithm, we have

·0000052 : ·0000049 :: ·1 : what is to be added to 8340·7;

therefore we must add

$$\frac{\cdot 0000049 \times \cdot 1}{\cdot 0000052}, \text{ or, } \frac{49 \times \cdot 1}{52} \text{ or } \cdot 094;$$

therefore number required is 8340·794.

If the given logarithm be negative, as −(2·1401355), we can change it into another of which the characteristic only is negative, thus $\bar{3}\cdot 8598645$, and we can find the number corresponding to this logarithm, as before. The number required is ·008340794.

Examples.—XLII.

(1) Given $\log 12954 = 4\cdot 1124039$,
$\log 12955 = 4\cdot 1124374$,

find the number whose logarithm is 4·112431.

(2) Given $\log 462\cdot 45 = 2\cdot 6650648$,
$\log 462\cdot 46 = 2\cdot 6650742$,

find the number whose logarithm is 3·6650657.

(3) Given $\log 34572 = 4\cdot 5387245$,
$\log 34573 = 4\cdot 5387371$,

find the number whose logarithm is 2·5387359.

(4) Given $\log 39375 = 4\cdot 5952206$,
$\log 39376 = 4\cdot 5952316$,

find the number whose logarithm is 5·5952282.

(5) Given $\log 3{\cdot}7159 = {\cdot}5690640$,
$\log 3{\cdot}7160 = {\cdot}5700757$,

find the number whose logarithm is 3·5700702.

(6) Given $\log 96461 = 4{\cdot}9843518$,
$\log 96462 = 4{\cdot}9843563$,

find the number whose logarithm is $\bar{3}{\cdot}9843542$.

(7) Given $\log 25725 = 4{\cdot}4103554$,
$\log 25726 = 4{\cdot}4103723$,

find the number whose logarithm is $\bar{7}{\cdot}4103720$.

(8) Given $\log 60195 = 4{\cdot}7795532$,
$\log 60196 = 4{\cdot}7795604$,

find the number whose logarithm is 2·7795607.

(9) Given $\log 10905 = 4{\cdot}0376257$,
$\log 10906 = 4{\cdot}0376655$,

find the number whose logarithm is 3·0376371.

(10) Given $\log 26201 = 4{\cdot}4183179$,
$\log 26202 = 4{\cdot}4183344$,

find the number whose logarithm is 2·4183314.

II. *On Tables of the Trigonometrical Ratios.*

162. We have explained in earlier parts of this treatise how to find the values of certain Trigonometrical Ratios exactly or approximately.

Thus we showed that $\sin 30^\circ = \frac{1}{2}$, that is, ·5 exactly.

Again, $\tan 60^\circ = \sqrt{3}$, that is, 1·73205 approximately.

Now the values of all the Trigonometrical Ratios in the first quadrant have been calculated and registered in tables. In some tables the angles succeed each other at intervals of 1″, in others at intervals of 10″, but in ordinary tables at intervals of 1′, and to the last-mentioned we shall refer.

These tables are commonly called Tables of *Natural* Sines, Cosines, &c. so as to distinguish them from the Tables of the *Logarithms* of the Sines, Cosines, &c. of which we shall hereafter treat.

We intend to explain, first, how we can determine the value of a Ratio that lies between the Ratios of two consecutive angles given in the tables, and secondly, how to determine the angle to which a given Ratio corresponds.

163. *To find the sine of a given angle.*

Suppose we want to determine the sine of 25°. 14′. 20″, having given

$$\sin 25^\circ.\,14' = \cdot 4263056 \text{ and } \sin 25^\circ.\,15' = \cdot 4265687.$$

$$\begin{array}{ll} \text{From } \sin 25^\circ.\,15' = & \cdot 4265687 \\ \text{Take } \sin 25^\circ.\,14' = & \underline{\cdot 4263056} \\ & \cdot 0002631 \text{ is the difference for } 1'. \end{array}$$

Now if we have to add ·0002631 to the sine of 25°. 14′ to obtain the sine of 25°. 15′, what must we add to the sine of 25°. 14′ to obtain the sine of 25°. 14′. 20″?

Assuming that an increase in the angle is proportional to an increase in the sine, we have

$$1' : 20'' :: \cdot 0002631 : \text{that which we have to add};$$

therefore we must add

$$\frac{20 \times \cdot 0002631}{60}, \text{ or, } \cdot 0000877;$$

$$\therefore \text{sine } 25^\circ.\,14'.\,20'' = \cdot 4263056 + \cdot 0000877$$
$$= \cdot 4263933.$$

164. *To find the cosine of a given angle.*

If we have to determine the cosine of $74^\circ.\,45'.\,40''$, having given

$$\cos 74^\circ.\,46' = \cdot 4263056 \text{ and } \cos 74^\circ.\,45' = \cdot 4265687,$$

we proceed thus:

$$\begin{aligned} \cos 74^\circ.\,45' &= \cdot 4265687 \\ \cos 74^\circ.\,46' &= \cdot 4263056 \end{aligned}$$

$\cdot 0002631$ is the *decrease* corresponding to $1'$,

observing that the cosine *decreases* as the angle increases from 0° to 90°.

Hence

$1' : 40'' :: \cdot 0002631$: what we have to take from $\cdot 4265687$;

therefore we must take away

$$\frac{40 \times \cdot 0002631}{60} = \cdot 0001754;$$

$$\begin{aligned} \therefore \cos 74^\circ.\,45'.\,40'' &= \cdot 4265687 - \cdot 0001754 \\ &= \cdot 4263933. \end{aligned}$$

Similar methods are to be taken to find the values of the other ratios, observing that the tangents and secants increase and the cotangents and cosecants decrease as the angle increases from 0° to 90°.

EXAMPLES.—XLIII.

(1) Given $\sin 42^\circ.\,15' = \cdot 6723668$,
$\sin 42^\circ.\,16' = \cdot 6725821$,
find $\sin 42^\circ.\,15'.\,16''$.

(2) Given $\sin 72^\circ = \cdot 9510565$,
$\sin 73^\circ = \cdot 9563048$,
find $\sin 72^\circ.\,14'$.

9—2

(3) Given $\sin 54^\circ = \cdot 8090170,$
$\sin 55^\circ = \cdot 8191520,$
find $\sin 54^\circ .\ 36'.$

(4) Given $\sin 87^\circ .\ 26' = \cdot 9989968,$
$\sin 87^\circ .\ 27' = \cdot 9990098,$
find $\sin 87^\circ .\ 26' .\ 15''.$

(5) Given $\sin 43^\circ .\ 14' = \cdot 6849711,$
$\sin 43^\circ .\ 15' = \cdot 6851830,$
find $\sin 43^\circ .\ 14' .\ 20''.$

(6) Given $\cos 41^\circ .\ 13' = \cdot 7522233,$
$\cos 41^\circ .\ 14' = \cdot 7520316,$
find $\cos 41^\circ .\ 13' .\ 26''.$

(7) Given $\tan 1^\circ .\ 22' = \cdot 0238573,$
$\tan 1^\circ .\ 23' = \cdot 0241484,$
find $\tan 1^\circ .\ 22' .\ 30''.$

(8) Given $\cot 35^\circ = 1 \cdot 428145,$
$\cot 36^\circ = 1 \cdot 376382,$
find $\cot 35^\circ .\ 53'.$

(9) Given $\sin 67^\circ .\ 22' = \cdot 9229865,$
$\sin 67^\circ .\ 23' = \cdot 9230984,$
find $\sin 67^\circ .\ 22' .\ 48'' \cdot 5.$

(10) Given $\cos 34^\circ .\ 12' = \cdot 8270806,$
$\cos 34^\circ .\ 13' = \cdot 8269170,$
find $\cos 34^\circ .\ 12' .\ 19'' \cdot 6.$

165. *To find the angle which corresponds to a given sine.*

Suppose the given sine to be ·5082784.

We find from the Tables

$$\begin{array}{rl} \text{sine } 30^\circ.\,33' = & \cdot 5082901 \\ \text{sine } 30^\circ.\,32' = & \underline{\cdot 5080396} \\ & \cdot 0002505 \text{ difference for } 1'. \end{array}$$

$$\begin{array}{rl} \text{given sine} = & \cdot 5082784 \\ \text{sine } 30^\circ.\,32' = & \underline{\cdot 5080396} \\ & \cdot 0002388 \end{array}$$

Hence if x be the number of seconds to be added to $30^\circ.\,32'$,

$$\cdot 0002505 : \cdot 0002388 :: 60 : x;$$

$$\therefore\ x = \frac{2388 \times 60}{2505} = \frac{9552}{167} = 57 \cdot 2 \text{ nearly};$$

$$\therefore \text{ the required angle is } 30^\circ.\,32'.\,57'' \cdot 2.$$

166. *To find the angle which corresponds to a given cosine.*

Suppose the given cosine to be ·5082784.

We find from the Tables

$$\begin{array}{rl} \cos 59^\circ.\,27' = & \cdot 5082901 \\ \cos 59^\circ.\,28'' = & \underline{\cdot 5080396} \\ = & \cdot 0002505 \text{ difference for } 1'. \end{array}$$

$$\begin{array}{rl} \cos 59^\circ.\,27' = & \cdot 5082901 \\ \text{given cosine} = & \underline{\cdot 5082784} \\ & \cdot 0000117 \end{array}$$

Hence if x be the number of seconds to be added to $59^\circ.\,27'$,

$$\cdot 0002505 : \cdot 0000117 = 60 : x;$$

$$\therefore\ x = \frac{117 \times 60}{2505} = 2 \cdot 8 \text{ nearly};$$

$$\therefore \text{ required angle is } 59^\circ.\,27'.\,2'' \cdot 8.$$

EXAMPLES.—XLIV.

(1) Given $$\begin{aligned}\sin 48^\circ.\,47' &= \cdot 7522233,\\ \sin 48^\circ.\,46' &= \cdot 7520316,\end{aligned}$$

find the angle of which the sine is $\cdot 752140$.

(2) Given $$\begin{aligned}\cos 2^\circ.\,34' &= \cdot 9989968,\\ \cos 2^\circ.\,33' &= \cdot 9990098,\end{aligned}$$

find the angle of which the cosine is $\cdot 999000$.

(3) Given $$\begin{aligned}\sin 43^\circ &= \cdot 6819984,\\ \sin 44^\circ &= \cdot 6946584,\end{aligned}$$

find the angle of which the sine is $\cdot 6849711$.

(4) Given $$\begin{aligned}\cos 32^\circ.\,31' &= \cdot 8432351,\\ \cos 32^\circ.\,32' &= \cdot 8430787,\end{aligned}$$

find the angle of which the cosine is $\cdot 8432$.

(5) Given $$\begin{aligned}\sin 24^\circ.\,11' &= \cdot 4096577,\\ \sin 24^\circ.\,12' &= \cdot 4099230,\end{aligned}$$

find the angle of which the sine is $\cdot 4097559$.

(6) Given $$\begin{aligned}\sec 82^\circ.\,22' &= 7 \cdot 528249,\\ \sec 82^\circ.\,23' &= 7 \cdot 552169,\end{aligned}$$

find the angle of which the secant is $7 \cdot 53$.

(7) Given $$\begin{aligned}\cos 53^\circ.\,7' &= \cdot 6001876,\\ \cos 53^\circ.\,8' &= \cdot 5999549,\end{aligned}$$

find the angle of which the cosine is $\cdot 6$.

(8) Given $$\begin{aligned}\operatorname{cosec} 25^\circ.\,3' &= 2 \cdot 36179,\\ \operatorname{cosec} 25^\circ.\,4' &= 2 \cdot 36029,\end{aligned}$$

find the angle of which the cosecant is $2 \cdot 361$.

(9) Given $\sin 73^\circ.\,44' = \cdot 9599684$,

$\sin 73^\circ.\,45' = \cdot 9600499$,

find the angle of which the sine is ·96.

(10) Given $\tan 77^\circ.\,19' = 4\cdot 44338$,

$\tan 77^\circ.\,20' = 4\cdot 44942$,

find the angle of which the tangent is 4·4.

III. *On Tables of Logarithms of Trigonometrical Ratios.*

167. The Trigonometrical Ratios being numbers have logarithms that correspond to them, and these logarithms are in practice much more useful than the numbers themselves.

Now since the sines and cosines of all angles and the tangents of angles less than 45° are *less than unity*, the logarithms of these sines, cosines and tangents are negative. In order to avoid the inconvenience of printing negative characteristics, the logarithms of all the Ratios given in the tables are *increased by* 10. The numbers thus registered are called *The Tabular Logarithms* of the sine, cosine, &c., and they are denoted by the symbol L, that is, $L \sin A$ denotes the tabular logarithm of the sine of A.

When the value of any one of these Tabular Logarithms is given we must take away 10 from it to obtain the true value of the logarithm in question: thus

$L \sin 25^\circ$ is set down in the tables as $9\cdot 6259483$,

and the true value of the logarithm of the sine of 25° is therefore

$$9\cdot 6259483 - 10, \text{ that is, } -\cdot 3740517,$$

or we might adopt the usual logarithmic notation of Art. 149, and say

$$\log.\sin 25^\circ = \bar{1}\cdot 6259483.$$

The Tables to which we refer are calculated for all angles in the first quadrant at intervals of 1′.

168. *To find the logarithmic sine of an angle not exactly given in the tables.*

Suppose we have to find $L\sin 6^\circ. 32'. 37''$.

$$\begin{aligned} L\sin 6^\circ. 33' &= 9{\cdot}0571723 \\ L\sin 6^\circ{\cdot} 32' &= \underline{9{\cdot}0560706} \\ &\quad {\cdot}0011017 \text{ difference for } 1'. \end{aligned}$$

Then if x be the number to be *added* to 9·0560706 to give us $L\sin 6^\circ. 32'. 37''$, we have

$$60 : 37 = {\cdot}0011017 : x;$$

$$\therefore\ x = \frac{{\cdot}0011017 \times 37}{60} = {\cdot}0006794, \text{ nearly};$$

$$\begin{aligned} \therefore\ L\sin 6^\circ. 32'. 37'' &= 9{\cdot}0560706 + {\cdot}0006794 \\ &= 9{\cdot}05675. \end{aligned}$$

169. *To find the logarithmic cosine of an angle not exactly given in the tables.*

Suppose we have to find $L\cos 83^\circ. 27'. 23''$.

$$\begin{aligned} L\cos 83^\circ. 27' &= 9{\cdot}0571723 \\ L\cos 83^\circ. 28' &= \underline{9{\cdot}0560706} \\ &\quad {\cdot}0011017 \text{ difference for } 1'. \end{aligned}$$

Then, if x be the number to be *subtracted* from 9·0571723 to give us $L\cos 83^\circ. 27'. 23''$, we have

$$60 : 23 = {\cdot}0011017 : x;$$

$$\therefore\ x = \frac{{\cdot}0011017 \times 23}{60} = {\cdot}0004223 \text{ nearly};$$

$$\begin{aligned} \therefore\ L\cos 83^\circ. 27'. 23'' &= 9{\cdot}0571723 - {\cdot}0004223 \\ &= 9{\cdot}05675. \end{aligned}$$

Similar methods must be taken to find the Tabular Logarithms of the other Trigonometrical Ratios: it being remembered that the tangent and secant *increase*, and the cotangent and cosecant *decrease*, as we pass from 0° to 90°.

EXAMPLES.—XLV.

(1) Given $L \sin 55^\circ.\, 33' = 9{\cdot}9162539,$
$L \sin 55^\circ.\, 34' = 9{\cdot}9163406,$
find $L \sin 55^\circ.\, 33'.\, 54''.$

(2) Given $L \sin 29^\circ.\, 25' = 9{\cdot}6912205,$
$L \sin 29^\circ.\, 26' = 9{\cdot}6914445,$
find $L \sin 29^\circ.\, 25'.\, 2''.$

(3) Given $L \cos 37^\circ.\, 28' = 9{\cdot}8996604,$
$L \cos 37^\circ.\, 29' = 9{\cdot}8995636,$
find $L \cos 37^\circ.\, 28'.\, 36''.$

(4) Given $L \sin 54^\circ.\, 13' = 9{\cdot}9091461,$
$L \sin 54^\circ.\, 14' = 9{\cdot}9092371,$
find $L \sin 54^\circ.\, 13'.\, 19''.$

(5) Given $L \tan 27^\circ.\, 42' = 9{\cdot}7201690,$
$L \tan 27^\circ.\, 43' = 9{\cdot}7204759,$
find $L \tan 27^\circ.\, 42'.\, 34''.$

(6) Given $L \tan 5^\circ.\, 13' = 8{\cdot}9604728,$
$L \tan 5^\circ.\, 14' = 8{\cdot}9618659,$
find $L \tan 5^\circ.\, 13'.\, 23''.$

(7) Given $L \cot 3^\circ.\, 37' = 11{\cdot}1992368,$
$L \cot 3^\circ.\, 38' = 11{\cdot}1972347,$
find $L \cot 3^\circ.\, 37'.\, 50''.$

(8) Given $L \sin 39^\circ.\, 25' = 9{\cdot}8027431,$
$L \sin 39^\circ.\, 26' = 9{\cdot}8028968,$
find $L \sin 39^\circ.\, 25'.\, 10''.$

(9) Given $L \sin 70^\circ.\ 34' = 9{\cdot}9745252,$

$L \sin 70^\circ.\ 35' = 9{\cdot}9745697,$

find $L \sin 70^\circ.\ 34'.\ 17''.$

(10) Given $L \cos 88^\circ.\ 54' = 8{\cdot}2832434,$

$L \cos 88^\circ.\ 55' = 8{\cdot}2766136,$

find $L \cos 88^\circ.\ 54'.\ 16''.$

170. *To find the angle which corresponds to a given Tabular Logarithmic Sine.*

Let the given L sine be 8·878594.

We find from the tables

$$\begin{array}{r} L \sin 4^\circ.\ 21' = 8{\cdot}8799493 \\ L \sin 4^\circ.\ 20' = 8{\cdot}8782854 \\ \hline {\cdot}0016639 \end{array} \text{ difference for } 1'.$$

$$\begin{array}{r} \text{given } L \sin = 8{\cdot}8789540 \\ L \sin 4^\circ.\ 20' = 8{\cdot}8782854 \\ \hline {\cdot}0006686 \end{array}$$

Hence if x be the number of seconds to be added to $4^\circ.\ 20'$,

$${\cdot}0016639 : {\cdot}0006686 = 60 : x;$$

$$\therefore\ x = \frac{6686 \times 60}{16639} = 24 \text{ nearly};$$

$$\therefore \text{ required angle is } 4^\circ.\ 20'.\ 24''.$$

171. *To find the angle which corresponds to a given Tabular Logarithmic Cosine.*

Let the given L cosine be 8·878954.

We find from the tables

$$\begin{array}{r} L \cos 85^\circ.\ 39' = 8{\cdot}8799493 \\ L \cos 85^\circ.\ 40' = 8{\cdot}8782854 \\ \hline {\cdot}0016639 \end{array} \text{ difference for } 1'.$$

$$\begin{array}{rl} L \cos 85^\circ.\, 39' = & 8{\cdot}8799493 \\ \text{given } L \text{ cosine} = & \underline{8{\cdot}8789540} \\ & {\cdot}0009953 \end{array}$$

Hence if x be the number of seconds to be added to $85^\circ.\, 39'$,

$$\cdot 0016639 : \cdot 0009953 = 60 \;;\; x;$$

$$\therefore x = \frac{9953 \times 60}{16639} = 35{\cdot}8 \text{ nearly};$$

$$\therefore \text{ required angle is } 85^\circ.\, 39'.\, 35''{\cdot}8.$$

Examples.—XLVI.

(1) Given $L \sin 14^\circ.\, 24' = 9{\cdot}3956581$,
$L \sin 14^\circ.\, 25' = 9{\cdot}3961499$,

find the angle whose $L \sin$ is $9{\cdot}3959449$.

(2) Given $L \sin 54^\circ.\, 13' = 9{\cdot}9091461$,
$L \sin 54^\circ.\, 14' = 9{\cdot}9092371$,

find the angle whose $L \sin$ is $9{\cdot}9091760$.

(3) Given $L \sin 71^\circ.\, 40' = 9{\cdot}9773772$,
$L \sin 71^\circ.\, 41' = 9{\cdot}9774191$,

find the angle whose $L \sin$ is $9{\cdot}9773897$.

(4) Given $L \cos 29^\circ.\, 25' = 9{\cdot}9400535$,
$L \cos 29^\circ.\, 26' = 9{\cdot}9399823$,

find the angle whose $L \cos$ is $9{\cdot}9400512$.

(5) Given $L \tan 30^\circ.\, 50' = 9{\cdot}7759077$,
$L \tan 30^\circ.\, 51' = 9{\cdot}7761947$,

find the angle whose $L \tan$ is $9{\cdot}7760397$.

(6) Given $L \cot 86^\circ.\, 32' = 8{\cdot}7823199$,
$L \cot 86^\circ.\, 33' = 8{\cdot}7802218$,

find the angle whose $L \cot$ is $8{\cdot}7814643$.

(7) Given $L \sin 24^\circ.\ 8' = 9{\cdot}6115762,$
$L \sin 24^\circ.\ 9' = 9{\cdot}6118580,$

find the angle whose $L \sin$ is $9{\cdot}6117876$.

(8) Given $L \tan 11^\circ.\ 39' = 9{\cdot}3142468,$
$L \tan 11\ .\ 40' = 9{\cdot}3148851,$

find the angle whose $L \tan$ is $9{\cdot}3148011$.

(9) Given $L \operatorname{cosec} 46\ .\ 23' = 10{\cdot}1402787,$
$L \operatorname{cosec} 46\ .\ 24' = 10{\cdot}1401584,$

find the angle whose $L \operatorname{cosec}$ is $10{\cdot}1402567$.

(10) Given $L \sec 29^\circ.\ 54' = 10{\cdot}0620326,$
$L \sec 29^\circ.\ 55' = 10{\cdot}0621053,$

find the angle whose $L \sec$ is $10{\cdot}062068557$.

CHAPTER XIV.

ON THE RELATIONS BETWEEN THE SIDES OF A TRIANGLE AND THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF THE TRIANGLE.

172. A TRIANGLE is composed of six parts, three sides and three angles.

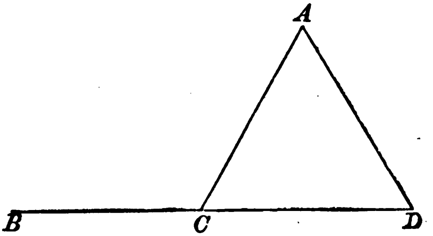
Three of these parts being given, one at least of the three being a side, we can generally determine the other three parts.

If only the three angles be given, we cannot determine the sides, because an infinite number of triangles may be constructed with the three angles of the one equal to the three angles of the other, each to each.

173. We shall denote the angles of a triangle by the letters A, B, C; the sides respectively opposite to them by the letters a, b, c.

The student must remember the results established in Art. 97,

$$\sin(180^\circ - A) = \sin A,$$
$$\cos(180^\circ - A) = -\cos A.$$



Thus, if ACB be an exterior angle of the triangle ACD,

$$\sin ACB = \sin ACD,$$
$$\cos ACB = -\cos ACD.$$

The results of Art. 62 are also frequently employed in this and the next Chapters.

174. *To shew that in any triangle*

$$c = a \,.\, \cos B + b \,.\, \cos A.$$

Fig. 1.

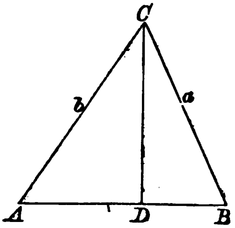
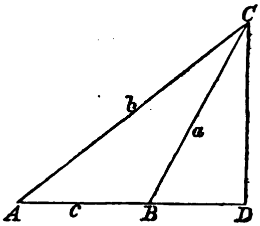


Fig. 2.



Let A, B be any two angles of the triangle ABC, and as one of them must be acute, let it be A.

Then according as B is acute or obtuse, draw CD at right angles to AB or to AB produced.

Then, in fig. 1,

$$\begin{aligned} c &= AD + DB \\ &= AC \,.\, \cos A + BC \,.\, \cos B \\ &= b \,.\, \cos A + a \,.\, \cos B\,; \end{aligned}$$

and in fig. 2,

$$\begin{aligned} c &= AD - DB \\ &= AC \,.\, \cos A - CB \,.\, \cos CBD \\ &= b \,.\, \cos A + a \,.\, \cos B, \text{ since } \cos CBD = -\cos ABC. \end{aligned}$$

If the angle at B be a right angle the theorem holds good, for then $\cos B = \cos 90^\circ = 0$, and $\therefore\ a \,.\, \cos B = 0$, and we have

$$c = b \,.\, \cos A.$$

175. *To shew that in every triangle the sides are proportional to the sines of the opposite angles.*

Fig. 1.

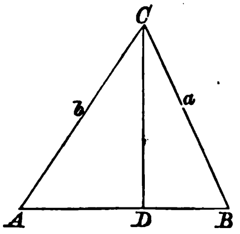
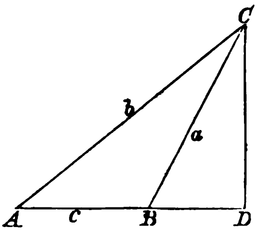


Fig. 2.



Let A, B be any two angles of the triangle ABC, and as one of them must be an acute angle, let it be A.

Then, according as B is acute or obtuse, draw CD at right angles to AB or to AB produced.

Then in fig. 1,

$$\sin A = \frac{CD}{b},$$

$$\sin B = \frac{CD}{a};$$

$$\therefore \frac{\sin A}{\sin B} = \frac{CD}{b} \div \frac{CD}{a} = \frac{CD}{b} \times \frac{a}{CD} = \frac{a}{b};$$

and in fig. 2,

$$\sin A = \frac{CD}{b},$$

$$\sin B = \sin(180^\circ - B) = \sin CBD = \frac{CD}{a};$$

$$\therefore \frac{\sin A}{\sin B} = \frac{CD}{b} \div \frac{CD}{a} \quad \frac{CD}{b} \times \frac{a}{CD} = \frac{a}{b}.$$

If the angle at B be a right angle the theorem still holds good, for then

$$\sin A = \frac{a}{b},$$

$$\sin B = 1;$$

$$\therefore \frac{\sin A}{\sin B} = \frac{a}{b}.$$

Similarly it may be shewn that in any triangle

$$\frac{\sin A}{\sin C} = \frac{a}{c} \text{ and } \frac{\sin B}{\sin C} = \frac{b}{c},$$

and therefore we conclude that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

176. *To express the cosine of an angle of a triangle in terms of the sides.*

Fig. 1.

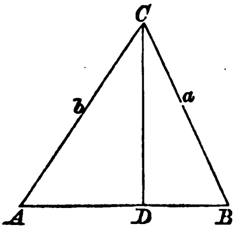
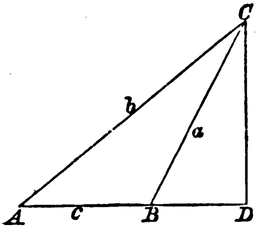


Fig. 2.



Let A, B be any two angles of the triangle ABC, and as one of them must be acute, let it be A.

Then, according as B is acute or obtuse, draw CD at right angles to AB or to AB produced.

Now, in fig. 1, by Euclid II. 13,

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BD,$$

or

$$b^2 = c^2 + a^2 - 2c \cdot BD.$$

Now, from the right-angled triangle BCD,

$$\frac{BD}{BC} = \cos B, \text{ or } \frac{BD}{a} = \cos B, \text{ or, } BD = a \cdot \cos B;$$

$$\therefore\ b^2 = c^2 + a^2 - 2ac \cdot \cos B.$$

Again, in fig. 2, by Euclid II. 12,

$$AC^2 = AB^2 + BC^2 + 2AB \cdot BD,$$

or

$$b^2 = c^2 + a^2 + 2c \cdot BD.$$

Now $\dfrac{BD}{BC} = \cos CBD = -\cos(180^\circ - CBD) = -\cos B$;

$$\therefore BD = -BC \,.\, \cos B, \text{ or } BD = -a \,.\, \cos B;$$

$$\therefore b^2 = c^2 + a^2 - 2ac \,.\, \cos B.$$

Hence, in each case,

$$2ac \,.\, \cos B = c^2 + a^2 - b^2,$$

or

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}.$$

So also

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

If B is a right angle, $\cos B = 0$, and the theorem still holds true, for then

$$b^2 = a^2 + c^2.$$

177. *To shew that*

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b} \,.\, \cot\frac{C}{2}.$$

Since

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2\cos\frac{A+B}{2} \,.\, \sin\frac{A-B}{2}}{2\sin\frac{A+B}{2} \,.\, \cos\frac{A-B}{2}} \quad \text{(Art. 116)}$$

$$= \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$$

$$= \frac{\tan\frac{A-B}{2}}{\cot\frac{C}{2}}, \quad \text{since } \frac{A+B}{2} = 90^\circ - \frac{C}{2};$$

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b} \,.\, \cot\frac{C}{2}.$$

178. If $s=\frac{a+b+c}{2}$, we can prove the following results:

$$(1)\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$(2)\quad \cos\frac{A}{2}=\sqrt{\frac{s.(s-a)}{bc}}.$$

The method of proof will be given in the next two articles.

179. First, to shew that

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Since $\cos A=1-2\sin^2\frac{A}{2}$,

$$2\sin^2\frac{A}{2}=1-\cos A$$

$$=1-\frac{b^2+c^2-a^2}{2bc}\quad \text{(Art. 176)}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc}.$$

Now if $s=\frac{a+b+c}{2}$, $2s=a+b+c$;

$$\therefore a+b-c=2s-2c \text{ and } a-b+c=2s-2b;$$

$$\therefore\ 2\sin^2\frac{A}{2}=\frac{2(s-c)\,.\,2(s-b)}{2bc};$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)\,.\,(s-c)}{bc}}.$$

We must take the positive sign with the root-symbol, because A being an angle of a triangle must be less than 180°, and therefore $\frac{A}{2}$ less than 90°, and consequently $\sin\frac{A}{2}$ is positive.

180. Next, to shew that

$$\cos\frac{A}{2}=\sqrt{\frac{s\,.\,(s-a)}{bc}}.$$

Since $$\cos A=2\cos^2\frac{A}{2}-1;$$

$$\therefore\ 2\cos^2\frac{A}{2}=1+\cos A$$

$$=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}.$$

Now, if $$s=\frac{b+c+a}{2},$$

$$b+c+a=2s \text{ and } b+c-a=2s-2a;$$

$$\therefore\ 2\cos^2\frac{A}{2}=\frac{2s\,.\,2(s-a)}{2bc};$$

$$\therefore\ \cos^2\frac{A}{2}=\frac{s\,.\,(s-a)}{bc};$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s\,.\,(s-a)}{bc}}.$$

10—2

181. From the preceding articles we may at once derive two other formulæ :

(1) $$\sin A = 2\sin\frac{A}{2}.\cos\frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{s.(s-a)}{bc}}$$

$$= \frac{2}{bc}.\sqrt{s.(s-a)(s-b)(s-c)},$$

(2) $$\tan\frac{A}{2} = \sin\frac{A}{2} \div \cos\frac{A}{2}$$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s.(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s.(s-a)}}.$$

EXAMPLES.—XLVII.

Prove the following relations when A, B, C are the angles of a triangle :

(1) $\sin(A+B) = \sin C$.

(2) $\cos(A+B) = -\cos C$.

(3) $\sin\frac{A+B}{2} = \cos\frac{C}{2}$.

(4) $\cos\frac{A+B}{2} = \sin\frac{C}{2}$.

(5) $\tan\frac{A+B}{2} = \cot\frac{C}{2}$.

(6) $\cot\frac{A+B}{2} = \tan\frac{C}{2}$.

Many other relations may be established by the use of the important formulæ explained in Art. 116, and the set of examples just given.

Thus, to shew that, if $A + B + C = 180°$,

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2},$$

we proceed thus,

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2} \quad \text{(Art. 116)}$$

$$= 2 \cos \frac{C}{2} \cdot \cos \frac{A-B}{2};$$

$$\therefore\ \sin A + \sin B + \sin C = 2 \cos \frac{C}{2} \cdot \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cdot \cos \frac{C}{2}$$

$$= 2 \cos \frac{C}{2} \left(\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right)$$

$$= 2 \cos \frac{C}{2} \left(2 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \right) \quad \text{(Art. 116)}$$

$$= 4 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}.$$

EXAMPLES.—XLVIII.

If A, B, C be the angles of a triangle, prove the following relations:

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

(2) $4 \sin A . \sin B . \sin C = \sin(-A + B + C) + \sin(A - B + C) + \sin(A + B - C).$

(3) $$\frac{\cot \frac{A}{2} + \cot \frac{C}{2}}{\cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{\sin B}{\sin A}.$$

(4) $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$

(5) $\cot A . \cot B + \cot A . \cot C + \cot B . \cot C = 1.$

(6) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} . \cot \frac{B}{2} . \cot \frac{C}{2}.$

(7) $4 \cos A . \cos B . \cos C = -(1 + \cos 2A + \cos 2B + \cos 2C).$

(8) $\cos A + \cos B + \cos C = 4 \sin \frac{A}{2} . \sin \frac{B}{2} . \sin \frac{C}{2} + 1.$

(9) $4 \sin A \,.\, \cos B \,.\, \cos C = - \sin 2A + \sin 2B + \sin 2C.$

(10) $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \,.\, \sin \frac{B}{2} \,.\, \cos \frac{C}{2}.$

(11) $\sin 2A + \sin 2B - \sin 2C = 4 \sin C \,.\, \cos A \,.\, \cos B.$

(12) $\cos A + \cos B - \cos C = 4 \cos \frac{A}{2} \,.\, \cos \frac{B}{2} \,.\, \sin \frac{C}{2} - 1.$

(13) $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \,.\, \sin \frac{B}{2} \,.\, \sin \frac{C}{2}.$

(14) $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \,.\, \sin \frac{B}{2} \,.\, \sin \frac{C}{2}.$

1. In a right-angled triangle where C is the right angle prove that

(1) $b + c = a \,.\, \cot \frac{A}{2}.$ (2) $2 \operatorname{cosec} 2A \,.\, \cot B = \frac{c^2}{b^2}.$

(3) $\sin \frac{B}{2} = \sqrt{\left(\frac{c-a}{2c}\right)}.$ (4) $\cos \frac{B}{2} = \sqrt{\left(\frac{a+c}{2c}\right)}.$

(5) $\frac{\cos 2B - \cos 2A}{\sin 2A} =$ ta[illegible]n B.

(6) $\tan 2A - \sec 2B = \frac{b+c}{b-c}.$

(7) $(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 4 \sin^2 \frac{C}{2}.$

(8) $\sec 2A = \frac{c^2}{b^2 - a^2}.$ (9) $abc = a^3 \,.\, \cos A + b^3 \,.\, \cos B.$

(10) $\cot (B - A) + \cot 2 \left(A + \frac{C}{2}\right) = 0.$

2. In any triangle prove the following relations:

(1) $\frac{\sin A - \sin B}{a - b} = \frac{\sin C}{c}.$ (2) $\frac{\sin (A - B)}{\sin C} = \frac{a^2 - b^2}{c^2}.$

(3) $\tan A = \frac{a \,.\, \sin C}{b - a \cos C}.$ (4) $\cot A = \frac{c}{a} \,.\, \operatorname{cosec} B - \cot B.$

(5) $a+b+c=(a+b)\cos C+(a+c)\cos B+(b+c)\cos A.$

(6) $(a+b)\sin\dfrac{C}{2}=c\,.\,\cos\dfrac{A-B}{2}.$

(7) $(a-b)\cos\dfrac{C}{2}=c\,.\,\sin\dfrac{A-B}{2}.$

(8) $\dfrac{\tan B}{\tan C}=\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}.$ (9) $c=a(\cos B+\sin B\,.\,\cot A).$

(10) $a^2+b^2+c^2=2(ab\,.\,\cos C+ac\,.\,\cos B+bc\,.\,\cos A).$

(11) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\,.\,\cos B\,.\,\cos C=1.$

(12) $\sin\dfrac{A-B}{2}=\dfrac{a-b}{c}\,.\,\cos\dfrac{C}{2}.$

(13) $\cos A+\cos B=\dfrac{a+b}{c}\,.\,2\sin^2\dfrac{C}{2}.$

(14) $a^2\sin A+ab\,.\,\sin B+ac\,.\,\sin C=(a^2+b^2+c^2)\sin A.$

(15) $\cot\dfrac{A}{2}:\cot\dfrac{B}{2}=b+c-a:a+c-b.$

(16) $\cot\dfrac{A}{2}\,.\,\cot\dfrac{B}{2}=\dfrac{a+b+c}{a+b-\text{[illegible]}}$

(17) $a\,.\,\sin(B-C)+b\,.\,\sin(C\,\text{[illegible]}\,A)+c\,.\,\sin(A-B)=0.$

3. If a, b, c be in arithmetical progression, shew that

$$\sin\left(A+\frac{B}{2}\right)=2\sin\frac{B}{2}.$$

4. ABC is a triangle inscribed in a circle and a point P is taken on the arc BC: shew that

$$PA\,.\,\sin A=PB\,.\,\sin B+PC\,.\,\sin C.$$

5. If ABC be a triangle and AD be drawn at right angles to BC, shew that

$$AD=\frac{b^2\sin C+c^2\sin B}{b+c}.$$

6. The sides of a triangle being 4, 9, 12, shew that the length of the line bisecting the angle between the two shorter sides is $2\frac{4}{13}$.

7. If $\sin A = 2\cos B . \sin C$, shew that the triangle is isosceles.

8. If $\cos A . \cos B . \sin C = \dfrac{\sin A + \sin B}{\sec A + \sec B}$, shew that $C = 90^\circ$.

9. If $\sin^2 A = \sin^2 B + \sin^2 C$, shew that $A = 90^\circ$.

10. If $\dfrac{a^3 + b^3 + c^3}{a + b + c} = c^2$ and also $\sin A . \sin B = \sin^2 C$, shew that the triangle is right angled.

11. If $C = 120^\circ$, shew that $c^2 = a^2 + ab + b^2$.

12. If CD bisect the angle C and meet AB in D, shew that

$$\tan ADC = \frac{a+b}{a-b}\tan\frac{C}{2}.$$

13. If CD bisect AB, shew that

$$CD^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}.$$

CHAPTER XV.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

182. LET a, b, c be the sides of a triangle and A, B, C the angles opposite to them. Of these *six* elements which present themselves in every triangle *three* must be known in order that we may determine the others, and one of these three must be *a side.*

183. The three angles of every triangle are together equal to two right angles: that is,

$$A + B + C = 180^\circ.$$

Hence, if two of the angles be known, the third will be known also.

184. Several of the results obtained in Chap. XIV. are to be carefully remembered, and especial attention must be given to the following formulæ, established in Arts. 175, 176, 177:

I. The Sine-rule, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

II. The Cosine-rule, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

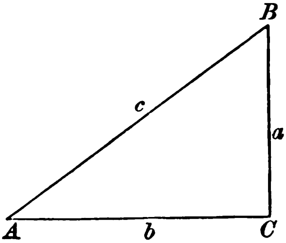
III. The Tangent-rule, $\tan \dfrac{A - B}{2} = \dfrac{a - b}{a + b} \cdot \cot \dfrac{C}{2}$.

185. We may now proceed to explain the method of solving right-angled triangles.

We shall denote the right angle by C. Then we may have the following data:

(1) Two sides and an angle,
(2) Two angles and a side.

186. First, when two sides and an angle are given, as b, c, C.



When two sides of a right-angled triangle are known, we can determine the third side; thus, in this case,

$$a = \sqrt{c^2 - b^2},$$

and so a is determined.

Next, $\sin A = \frac{a}{c}$, from which we can find A.

Lastly, $B = 180^\circ - (A + C)$, and so B is determined.

187. Next, when two angles and a side are given, as c, A, C.

First, $\frac{a}{c} = \sin A$, from which we can find a.

Next, $\frac{b}{c} = \cos A$, from which we can find b.

Lastly, $B = 180^\circ - (A + C)$, from which we can find B.

188. If A be one of the angles of a triangle and the value of $\sin A$ be given, we cannot determine the value of A without previously knowing whether A is an *acute* or an *obtuse* angle.

Thus, suppose we know that $\sin A = \frac{1}{2}$, one value of A which satisfies this equation is 30°, but another value of A which satisfies the equation is 150°, for since the sine of an angle is equal to the sine of the supplement of the angle,

$$\sin 150^\circ = \sin (180^\circ - 150^\circ) = \sin 30^\circ.$$

In a right-angled triangle A, when not the right angle, *must* be acute, and so in the cases we have considered no ambiguity can occur.

In oblique-angled and acute-angled triangles we shall find only one case in which we cannot determine with certainty the value of A from the known value of $\sin A$.

189. We shall now give some Examples of the practical application of the methods of solution described in the preceding Articles.

To take the first of the two cases, suppose we have the following data:

$$b = 5, \quad c = 13, \quad C = 90^\circ.$$

Then $$a = \sqrt{c^2 - b^2} = \sqrt{169 - 25} = \sqrt{144} = 12.$$

and $$\sin A = \frac{a}{c} = \frac{12}{13} = \cdot 9230769.$$

Now from the tables we find

$$\sin 67^\circ.\ 22' = \cdot 9229865,$$
$$\sin 67^\circ.\ 23' = \cdot 9230984.$$

And hence, by the method explained in Art. 168, we find the value of A to be $67^\circ.\ 22'.\ 48''\cdot 5$.

190. Again, to take the second case, suppose we have given

$$c = 25, \quad A = 60^\circ, \quad C = 90^\circ.$$

Then $$\frac{a}{c} = \sin A;$$

$$\therefore\ \frac{a}{25}=\frac{\sqrt{3}}{2},\quad \therefore\ a=\frac{25\sqrt{3}}{2}.$$

$$\frac{b}{c}=\cos A;$$

$$\therefore\ \frac{b}{25}=\frac{1}{2};\quad \therefore\ b=\frac{25}{2}.$$

Also, $B=180^\circ-(A+C)=180^\circ-150^\circ=30^\circ.$

EXAMPLES.—XLIX.

Solve the triangles referred to in the following examples by the use of natural sines, cosines, &c., C being a right angle.

(1) Given $b=3$, $c=5$, $\sin 53^\circ.\ 7'=\cdot 7998593$,
$\sin 53^\circ.\ 8'=\cdot 8000338$.

(2) Given $b=15$, $c=17$, $\sin 28^\circ.\ 4'=\cdot 4704986$,
$\sin 28^\circ.\ 5'=\cdot 4707553$.

(3) Given $b=21$, $c=29$, $\sin 43^\circ.\ 36'=\cdot 6896195$,
$\sin 43^\circ.\ 37'=\cdot 6898302$.

(4) Given $b=7$, $c=25$, $\cos 73^\circ.\ 44'=\cdot 2801083$,
$\cos 73^\circ.\ 45'=\cdot 2798290$.

(5) Given $b=33$, $c=65$, $\cos 59^\circ.\ 29'=\cdot 5077890$,
$\cos 59^\circ.\ 30'=\cdot 5075384$.

(6) Given $c=13$, $A=67^\circ.\ 22'.\ 48''\cdot 5$, $\sin 67^\circ.\ 22'=\cdot 9229865$,
$\sin 67^\circ.\ 23'=\cdot 9230984$.

(7) Given $c=41$, $A=77^\circ.\ 19'.\ 10''\cdot 6$, $\sin 77^\circ.\ 19'=\cdot 9755985$,
$\sin 77^\circ.\ 20'=\cdot 9756623$.

(8) Given $c=73$, $B=48^\circ.\ 53'.\ 16''\cdot 5$, $\cos 48^\circ.\ 53'=\cdot 6575944$,
$\cos 48^\circ.\ 54'=\cdot 6573752$.

(9) Given $c = 89$, $B = 64.0'.38''\cdot 8$, $\cos 64^\circ = \cdot 4383711$,
$\cos 64^\circ.1' = \cdot 4381097$.

(10) Given $a = 40$, $A = 77^\circ.19'.10''\cdot 6$, $\tan 77^\circ.19' = 4\cdot 4433769$,
$\tan 77^\circ.20' = 4\cdot 4494186$.

191. The process of solution by means of natural sines, cosines, &c. can only be applied with advantage to cases in which the measures of the sides are small numbers.

We proceed to shew how the use of logarithmic calculations assists us in the solution of triangles.

192. It must be observed that a formula is adapted to Logarithmic calculation only when it consists of the product or quotient of two or more numbers.

For instance, we derive no advantage from logarithms in finding c from the equation $c^2 = a^2 + b^2$, when a and b are given.

But if a and c be given, we can apply logarithms with advantage to find b from the equation

$$b^2 = c^2 - a^2;$$

for instance, if $a = 644$ and $c = 725$,

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= (c + a)(c - a) \\ &= 1369 \times 81; \end{aligned}$$

$$\therefore \log b^2 = \log(1369 \times 81);$$

$$\begin{aligned} \therefore 2\log b &= \log 1369 + \log 81 \\ &= 3\cdot 1364034 + 1\cdot 903090 \\ &= 5\cdot 0394934; \end{aligned}$$

$$\therefore \log b = 2\cdot 5197467;$$

$$\therefore b = 333.$$

193. Let $a = 644$, $c = 725$, $C = 90°$.

We first find $b = 333$, as explained in the preceding Article.

Then $$\sin A = \frac{a}{c};$$

$$\therefore \log \sin A = \log a - \log c.$$

Now $L \sin A$ is the true $\log \sin A$ increased by 10, Art. 167, hence we put here and in all similar cases $L \sin A - 10$ in place of $\log \sin A$.

Thus $$L \sin A - 10 = \log a - \log c,$$
$$L \sin A = 10 + 2{\cdot}8088859 - 2{\cdot}8603380$$
$$= 9{\cdot}9485479.$$

Now from the tables

$$L \sin 62°.\ 39' = 9{\cdot}9485189,$$
$$L \sin 62°.\ 40' = 9{\cdot}9485842.$$

Hence, by the method of Art. 168, we may find $A = 62°.\ 39'.\ 27''$ nearly; and therefore $B = 27°.\ 20'.\ 33''$.

EXAMPLES.—L.

Solve the triangles referred to in the following Examples by Logarithmic calculations, C being a right angle:

(1) Given $a = 104$, $c = 185$, $\log a = 2{\cdot}0170333$,
$\log c = 2{\cdot}2671717$, $\log 153 = 2{\cdot}1846914$,
$\log 289 = 2{\cdot}4608978$, $\log 81 = 1{\cdot}9084850$,
$L \sin 34°.\ 12' = 9{\cdot}7498007$, $L \sin 34°.\ 13' = 9{\cdot}7499866$.

(2) Given $a = 304$, $c = 425$, $\log a = 2{\cdot}4828736$,
$\log c = 2{\cdot}6283889$, $\log 297 = 2{\cdot}4727564$,
$\log 729 = 2{\cdot}8627275$, $\log 121 = 2{\cdot}0827854$,
$L \sin 45°.\ 40' = 9{\cdot}8544799$, $L \sin 45°\ 41' = 9{\cdot}8546033$.

(3) Given $a = 840$, $c = 841$, $\log a = 2{\cdot}9242793$,
$\log c = 2{\cdot}9247960$, $\log 41 = 1{\cdot}6127839$,
$\log 1681 = 3{\cdot}2255677$,
$L \sin 87^\circ.\,12' = 9{\cdot}9994812$, $L \sin 87^\circ.\,13' = 9{\cdot}9994874$.

(4) Given $a = 336$, $c = 625$, $\log a = 2{\cdot}5263393$,
$\log c = 2{\cdot}7958800$, $\log 527 = 2{\cdot}7218106$,
$\log 961 = 2{\cdot}9827234$, $\log 289 = 2{\cdot}4608978$,
$L \sin 32^\circ.\,31' = 9{\cdot}7304148$, $L \sin 32^\circ.\,32' = 9{\cdot}7306129$.

(5) Given $a = 1100$, $c = 1109$, $\log a = 3{\cdot}0413927$,
$\log c = 3{\cdot}0449315$, $\log 141 = 2{\cdot}1492191$,
$\log 2209 = 3{\cdot}3441957$, $\log 3 = {\cdot}4771213$,
$L \sin 82^\circ.\,41' = 9{\cdot}9964493$, $L \sin 82^\circ.\,42' = 9{\cdot}9964655$.

(6) Given $b = 195$, $c = 773$, $\log b = 2{\cdot}2900346$,
$\log c = 2{\cdot}8881795$, $\log 748 = 2{\cdot}8739016$,
$\log 968 = 2{\cdot}9858754$, $\log 578 = 2{\cdot}7619278$,
$L \cos 75^\circ.\,23' = 9{\cdot}4020048$, $L \cos 75^\circ.\,24' = 9.4015201$.

(7) Given $b = 273$, $c = 785$, $\log b = 2{\cdot}4361626$,
$\log c = 2{\cdot}8948697$, $\log 736 = 2{\cdot}8668778$,
$\log 1058 = 3{\cdot}0244857$, $\log 2 = {\cdot}3010300$,
$L \cos 69^\circ.\,38' = 9{\cdot}5416126$, $L \cos 69^\circ.\,39' = 9{\cdot}5412721$.

(8) Given $b = 609$, $c = 641$, $\log b = 2{\cdot}7846173$,
$\log c = 2{\cdot}8068580$,
$\log 1250 = 3{\cdot}0969100$, $\log 2 = {\cdot}3010300$,
$L \cos 18^\circ.\,10' = 9{\cdot}9777938$, $L \cos 18^\circ.\,11' = 9{\cdot}9777523$.

(9) Given $a = 276$, $b = 493$, $\log a = 2{\cdot}4409091$,
$\log b = 2{\cdot}6928469$,
$L \tan 29^\circ.\,14' = 9{\cdot}7479125$, $L \tan 29^\circ.\,15' = 9{\cdot}7482089$.

(10) Given $a = 396$, $b = 403$, $\log a = 2\cdot5976952$,

$\log b = 2\cdot6053050$,

$L \tan 44^\circ.\,29' = 9\cdot9921670$, $L \tan 44^\circ.\,30' = 9\cdot9924197$.

194. We shall now give a few Problems to illustrate the practical use of the methods of solution of triangles explained in this Chapter.

EXAMPLES.—LI.

(1) Having measured a distance of 220 feet in a direct horizontal line from the bottom of a steeple, the angle of elevation of its top was found to be 46°. 30′. Required the height of the steeple.

Given $\log 220 = 2\cdot3424227$, $\log \tan 46^\circ.\,30' = 10\cdot0227500$,

$\log 2\cdot31835 = \cdot3651727$.

(2) A river AC whose breadth is 200 feet runs at the foot of a tower CB, which subtends an angle BAC of 25°. 10′ at the edge of the bank.

Required the height of the tower, given

$\log 5 = \cdot6989700$, $L \tan 25^\circ.\,10' = 9\cdot6719628$,

$\log 9397 = 3\cdot9729928$.

(3) A person on the top of a tower, whose height is 50 feet, observes the angles of depression of two objects on the horizontal plane which are in the same straight line with the tower to be 30° and 45°. Find their distances from each other and from the observer.

(4) At 140 feet from the base of a tower, and on a level with the base the angle of elevation of the top was found to be 54°. 7′. Find the height of the tower, having given

$\tan 54^\circ.\,7' = 1\cdot399364$.

(5) A person observes the angle of elevation of a hill to be $32^\circ.\ 14'$, and on approaching 500 yards nearer, he observes it to be $63^\circ.\ 26'$. Find the height of the hill, having given

$$\tan 32^\circ.\ 14' = \cdot 63, \quad \tan 63^\circ.\ 26' = 1\cdot 998.$$

(6) A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane on which it stands. Find the sun's altitude, having given $\log 2 = \cdot 3010300$,

$$L \tan 63^\circ.\ 26' = 10\cdot 3009994, \quad L \tan 63^\circ.\ 27' = 10\cdot 3013153.$$

(7) A tower stands by a river. A person on the opposite bank finds its elevation to be 60°: he recedes 40 yards in a direct line from the tower, and then finds the elevation to be 50°. Find the breadth of the river, having given $\tan 50^\circ = 1\cdot 19$.

(8) A rope is fastened to the top of a building 60 feet high. The length of the rope is 109 feet. Find the angle at which it is inclined to the horizon.

Given $\quad \sin 33^\circ.\ 23' = \cdot 5502, \quad \sin 33^\circ.\ 24' = \cdot 55048.$

(9) A tower is 140 feet in height. At what angle must a rope be inclined to the horizon which reaches from the top of the tower to the ground, and is 222 feet in length?

Given $\quad \sin 39^\circ.\ 18' = \cdot 6333809, \quad \sin 39^\circ.\ 19' = \cdot 6336059.$

(10) A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 55° with a line drawn from his eye parallel to the horizon; receding backwards 30 feet, he then finds it to subtend an angle of 48°. Find the breadth of the river.

Given $\quad L \sin 7^\circ = 9\cdot 08589, \quad L \sin 35^\circ = 9\cdot 75859,$

$L \sin 48^\circ = 9\cdot 87107, \quad \log 3 = \cdot 47712, \quad \log 1\cdot 0493 = \cdot 02089.$

(11) Standing straight in front of the corner of a house which is 150 feet long, I observe that the length subtends an angle whose cosine is $\frac{1}{\sqrt{5}}$, and its height subtends an angle whose sine is $\frac{3}{\sqrt{34}}$; determine the height.

(12) Standing straight in front of one corner of a house, I find that its length subtends an angle whose tangent is 2, while its height subtends an angle whose tangent is $\frac{3}{5}$: the height of the house is 45 feet, find its length.

CHAPTER XVI.

ON THE SOLUTION OF TRIANGLES OTHER THAN RIGHT-ANGLED.

195. In the solution of triangles other than right-angled we meet with four distinct cases, the following being the data.

(1) The three sides a, b, c.

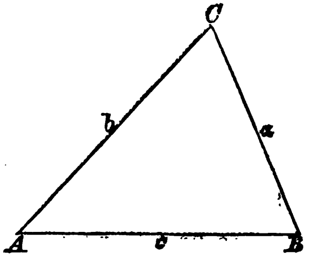
(2) Two angles and a side, as A, C, b.

(3) Two sides and the angle between them, as a, b, C.

(4) Two sides and an angle opposite one of them, as a, b, A.

These cases we shall discuss in the four following articles.

196. Case I. When a, b, c are given, we have the formula established in Art. 176,



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$ from which A can be found.

Or we can find A from the formula established in Art. 181,

$$\sin A = \frac{2}{bc} \cdot \sqrt{s \cdot (s-a)(s-b)(s-c)}.$$

Then, $\frac{\sin C}{\sin A} = \frac{c}{a}$, from which C can be found,

and $B = 180^\circ - (A + C)$, from which B can be found.

197. Case II. When A, C, b are given.

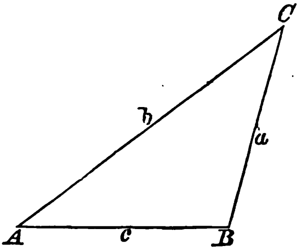
First, $B = 180^\circ - (A + C)$, from which we can find B.

Next, $\frac{a}{b} = \frac{\sin A}{\sin B}$, from which we can find a.

Lastly, $\frac{c}{b} = \frac{\sin C}{\sin B}$, from which we can find c.

198. Case III. When a, b, C are given, we may take the formula established in Art. 176,

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C, \text{ from which we can find } c.$$



Next, $\frac{\sin A}{\sin C} = \frac{a}{c}$, from which we can find A.

Lastly, $B = 180^\circ - (A + C)$, from which we can find B.

Or we may proceed to find A and B before we find c, thus: by the formula established in Art. 177,

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cdot \cot \frac{C}{2}.$$

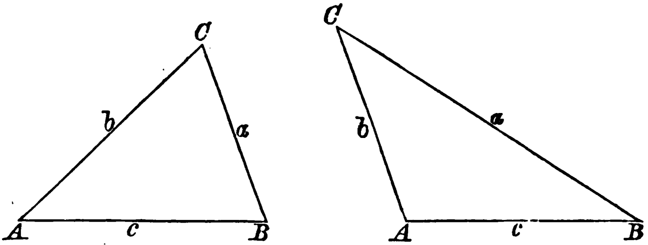
Thus we can find $A - B$.

And as we know that $A + B = 180° - C$,

we shall have two equations by which we may determine A and B.

Then we can find c from the equation $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$.

199. Case IV. When a, b, A are given



$\dfrac{\sin B}{\sin A} = \dfrac{b}{a}$, from which we have to determine B.

If we can find B, we have

$C = 180° - (A + B)$, from which we can find C,

and $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$, from which we can find c.

Thus the solution of this case depends on the possibility of determining B from the equation

$$\frac{\sin B}{\sin A} = \frac{b}{a} \quad (1).$$

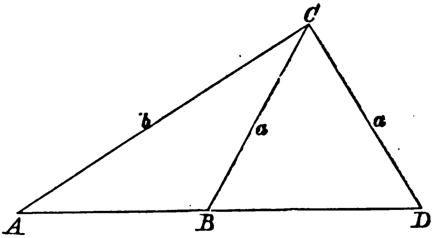
Now since we know a, b, A, we shall obtain from equation (1),

$\sin B =$ a known numerical quantity (2).

But, as has been explained in Art. 104, we cannot determine the value of B from a given value of $\sin B$, unless we know whether B is greater or less than 90°.

The only way in which we can tell whether the greater or the smaller value of B which satisfies equation (2) is to be taken, is by knowing *that* a *is greater than* b. In that case A is also greater than B, and therefore B must be less than 90°, otherwise $A+B$ would not be less than 180, which is impossible.

This, which is called *The Ambiguous Case,* may be illustrated in the following manner.



If from C we can draw a line CD equal to CB, to meet AB produced on the side of B, both the triangles ABC, ADC have the given parts a, b, A.

We can always draw $CD=CB$, *so long as* AC *is greater than* BC, for then a circle described with centre C and radius CB will cut AB produced in two points both on the same side of A.

200. In the Examples which we have now to put before the student, we shall follow the order of our remarks on the methods of solution available, and take the four cases successively.

201. Case I. Given the three sides a, b, c.

The formula $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ is not adapted to logarithmic calculation, but for *small* values of a, b, c we may use it to find A by aid of the table of natural cosines.

Thus, if $a=6$, $b=5$, $c=10$,

$$\cos A=\frac{25+100-36}{100}=\cdot 89.$$

And hence we find $A=27°.\ 7'.\ 36''$.

202. For logarithmic calculation we take the formula

$$\sin A = \frac{2}{bc}\sqrt{s\,.\,(s-a)\,.\,(s-b)\,.\,(s-c)},$$

where $$s = \frac{a+b+c}{2}.$$

Taking the same example, we have

$$\sin A = \frac{2}{50}\sqrt{\frac{21}{2}\times\frac{9}{2}\times\frac{11}{2}\times\frac{1}{2}},$$

$$\sin A = \frac{1}{25}\sqrt{\frac{2079}{16}};$$

$$\therefore\ L\sin A - 10 = \log 1 - \log 25 + \frac{1}{2}(\log 2079 - \log 16)$$

$$= 0 \ - 1\cdot3979400 + \frac{1}{2}(3\cdot3178545 - 1\cdot2041200);$$

$$\therefore\ L\sin A = 10 - 1\cdot3979400 + 1\cdot0568672$$
$$= 9\cdot6589272.$$

And hence we find by the Tables $A = 27^\circ.\,7'.\,36''$.

EXAMPLES.—LII.

Find A from the following data:

(1) Given $a = 37$, $b = 13$, $c = 40$, $\sin 67^\circ.\,22' = \cdot9229865$,
$\sin 67^\circ.\,23' = \cdot9230984$.

(2) Given $a = 101$, $b = 29$, $c = 120$, $\sin 43^\circ.\,36' = \cdot6896195$,
$\sin 43^\circ.\,37' = \cdot6898302$.

(3) Given $a = 37$, $b = 13$, $c = 30$, $\log 145 = 2\cdot1613680$,
$\log 324 = 2\cdot5105450$,
$L\sin 112^\circ.\,36' = 9\cdot9653006$, $L\sin 112^\circ.\,37' = 9\cdot9652480$.

(4) Given $a = 409$, $b = 241$, $c = 600$, $\log 723 = 2\cdot8591383$,
$\log 93636 = 4\cdot9714429$,
$L\sin 29^\circ.\,51' = 9\cdot6969947$, $L\sin 29^\circ.\,52' = 9\cdot6972148$.

203. When the sides are numbers containing several digits, we may employ with advantage the formula given in Art. 181

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s.(s-a)}},$$

from which we have

$$L\tan\frac{A}{2}-10=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\},$$

and then we may find $\tan\frac{B}{2}$ or $\tan\frac{C}{2}$ by means of the same logarithms, thus

$$L\tan\frac{B}{2}-10=\frac{1}{2}\{\log(s-a)+\log(s-c)-\log s-\log(s-b)\}.$$

For example, let $a=9459{\cdot}31$, $b=8032{\cdot}29$, $c=8242{\cdot}58$.

Then $s=12867{\cdot}09$, $s-a=3407{\cdot}78$,

$s-b=8434{\cdot}80$, $s-c=4624{\cdot}51$.

$$L\tan\frac{A}{2}-10=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}$$

$$=\frac{1}{2}\{3{\cdot}6843785+3{\cdot}6650657-4{\cdot}1094804-3{\cdot}5324716\}$$

$$=-{\cdot}1462539;$$

$$\therefore\ L\tan\frac{A}{2}=9{\cdot}8537461.$$

Whence $\frac{A}{2}=35^\circ.\,31'.\,47''{\cdot}7$,

and $A=71^\circ\ 3'.\,34''$.

Similarly we may find $B=53^\circ.\,26'$ and $C=55^\circ.\,30'.\,24''{\cdot}5$.

204. We have now discussed two of the formulæ given under Case I., and there remains the third formula

$$\frac{\sin C}{\sin A}=\frac{c}{a},$$

on which no comment is needed, except that expressed in logarithms it gives

$$L\sin C = L\sin A + \log c - \log a,$$

from which C may be determined.

205. Case II. Here we have no difficulty, and we shall merely give an example to illustrate the method of finding a from the formula

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \text{ when } b,\ A,\ B \text{ are known.}$$

Let $\qquad b = 40,\ A = 12^\circ.\,40',\ B = 77^\circ.\,10'.$

$$\begin{aligned}\text{Then } \log a &= \log b + L\sin A - L\sin B\\ &= 1{\cdot}6020600 + 9{\cdot}3409963 - 9{\cdot}9890137\\ &= {\cdot}9540426,\end{aligned}$$

whence $\qquad a = 9$ nearly.

206. Case III.

The first formula given,

$$c^2 = a^2 + b^2 - 2ab\,.\cos C,$$

is not adapted to logarithmic calculation.

We must take then, in all cases where a, b, c are not small integers, the formula

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\,.\cot\frac{C}{2},$$

for finding the values of A and B.

Suppose then we have given

$$a = 12{\cdot}96,\ b = 9{\cdot}78,\ C = 57^\circ.\,48'.\,32'',$$

we proceed thus:

$$a - b = 3{\cdot}18,\ a + b = 22{\cdot}74,\ \frac{C}{2} = 28^\circ.\,54'.\,16''.$$

$$\text{Then } L\tan\frac{A-B}{2} - 10 = \log(a-b) - \log(a+b) + L\cot\frac{C}{2} - 10,$$

$$L\tan\frac{A-B}{2} = \log 3{\cdot}18 - \log 22{\cdot}74 + L\cot 28^\circ.\,54'.\,16''$$
$$= {\cdot}5024271 - 1{\cdot}3567905 + 10{\cdot}2579579$$
$$= 9{\cdot}4035945.$$

Whence $\dfrac{A-B}{2} = 14^\circ.\,12'.\,46''.$

Also $\dfrac{A+B}{2} = 61^\circ.\,5'.\,44''$ $\left(\text{the complement of } \dfrac{C}{2}\right)$;

$$\therefore\ A = 75^\circ.\,18'.\,30'',$$
$$B = 46^\circ.\,52'.\,58''.$$

The other formulæ of this case require no special remark.

207. Case IV. The formulæ are here simple, and we have only to give instances of cases (1) ambiguous, (2) in which no ambiguity exists.

(1) To take a very simple case, suppose

$$a = 5,\ b = 6,\ A = 30^\circ.$$

Then $$\frac{\sin B}{\sin A} = \frac{b}{a};$$

$$\therefore\ \sin B = \frac{b}{a}.\sin A = \frac{6}{5}\times\frac{1}{2} = \frac{6}{10} = {\cdot}6.$$

Now from the tables we find ·6 to be the value of the sine of 36°. 52′. 12″, and since 143°. 7′. 48″ is the supplement of 36°. 52′. 12″, it also has ·6 for the value of its sine. Thus there is an ambiguity in the result.

Again, suppose $a = 178{\cdot}3,\ b = 145,\ B = 41^\circ.\,10'.$

Then $\sin A = \dfrac{a}{b}\sin B;$

$$\therefore\ L\sin A = \log a - \log b + L\sin B$$
$$= 2{\cdot}2511513 - 2{\cdot}1613680 + 9{\cdot}8183919$$
$$= 9{\cdot}9081752.$$

Hence $A = 54^\circ.\,2'.\,22''$ or $125^\circ.\,57'.\,38''.$

(2) Now if we change the values of the sides a and b we shall get

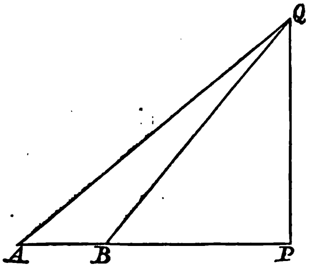
$$L \sin A = 9 \cdot 7286086\,;$$

$$\therefore\ A = 32^\circ.\ 21'.\ 54'',$$

and the supplement of A cannot belong to the proposed triangle, because if A were $147^\circ.\ 38'.\ 6''$, then, since B is greater than A, A and B would be together greater than 180°, which is impossible. So in this case there is no ambiguity.

208. We shall here insert two examples of some importance in the practical application of Trigonometry.

(1) To find the height of an object standing on a horizontal plane, when the base of the object is inaccessible.



Let PQ be a tower, of which the base P is inaccessible.

Measure a distance AB in the same horizontal plane with P.

Observe the angles of elevation QBP and QAP.

Then we can determine the height of the column, for

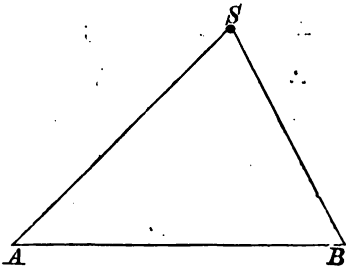
$$QP = QB\,.\,\sin QBP,$$

and $$QB = AB\,.\,\frac{\sin QAB}{\sin BQA}$$

$$= AB\,.\,\frac{\sin QAP}{\sin (QBP - QAP)}\,;$$

$$\therefore\ QP = AB\,.\,\sin QBP\,.\,\frac{\sin QAP}{\sin (QBP - QAP)}\,.$$

(2) To find the distance of a ship from the shore.



Let S be the position of the ship.

Measure AB, a straight line between two points on the shore.

Observe the angles SAB and SBA.

Then we can determine the distance of S from A, for

$$AS = AB \cdot \frac{\sin SBA}{\sin ASB}$$

$$= AB \cdot \frac{\sin SBA}{\sin (180^\circ - ASB)}$$

$$= AB \cdot \frac{\sin SBA}{\sin (SAB + SBA)}.$$

209. The following are miscellaneous examples on the principles laid down in this Chapter.

EXAMPLES.—LIII.

(1) If $b = 5780$, $c = 7639$, $B = 43^\circ.\ 8'$, find A and C, having given

$$\log 185{\cdot}9 = 2{\cdot}26928, \quad \log 13{\cdot}419 = 1{\cdot}12772,$$

$$L \cot 21^\circ.\ 34' = 10{\cdot}40312, \quad L \tan 19^\circ.\ 18'.\ 50'' = 9{\cdot}54468.$$

(2) If $A = 41^\circ.\ 13'.\ 22''$, $B = 71^\circ.\ 19'.\ 5''$, $a = 55$, find b, having given

$$\log 55 = 1{\cdot}7403627, \quad L \sin B = 9{\cdot}9764927, \quad L \sin A = 9{\cdot}8188779,$$

$$\log 79{\cdot}063 = 1{\cdot}8979775.$$

(3) If $B = 84^\circ.\ 47'.\ 38''$, $C = 41^\circ.\ 10'$, $c = 145$, find b, having given

$$\log 145 = 2\cdot1613680,\ L\sin 41^\circ.\ 10' = 9\cdot8183919,$$
$$L\sin 84^\circ.\ 47'.\ 38'' = 9\cdot9982047,\ \log 219\cdot37 = 2\cdot3411808.$$

(4) If $a = 567\cdot2341$, $b = 351\cdot9872$, $B = 31^\circ.\ 27'.\ 18''$, find A, having given

$$\log a = 2\cdot7537623,\ \log b = 2\cdot5465269,$$
$$L\sin B = 9\cdot7175280,\ L\sin 57^\circ.\ 14' = 9\cdot9247349,$$
$$L\sin 57^\circ.\ 15' = 9\cdot9248161.$$

(5) Wishing to know the height of an inaccessible hill I took the angle of elevation of its top to be 60°, I then measured 100 feet away from the hill and found the angle of elevation to be 45°. What is the height of the hill?

(6) Each of two ships, which are a mile apart, finds the angles subtended by the other ship and a fort to be respectively $35^\circ.\ 14'$ and $42^\circ.\ 12'$. Find the distance of each from the fort. Given $\sin 35^\circ.\ 14' = \cdot577$, $\sin 42^\circ.\ 12' = \cdot671$, $\sin 77^\circ.\ 26' = \cdot976$.

(7) Each of two ships, half a mile apart, finds the angles subtended by the other ship and a fort to be respectively $85^\circ.\ 15'$ and $83^\circ.\ 45'$. Find the distance of each from the fort. Given $\sin 85^\circ.\ 15' = \cdot9965$, $\sin 83^\circ.\ 45' = \cdot9940$, $\sin 11^\circ = \cdot1908$.

(8) Find the angle which a flag-staff 5 yards long and standing on the top of a tower two hundred yards high subtends at a point in the horizontal plane 100 yards from the base of the tower. Given $\tan 34'.\ 22'' = \cdot01$.

(9) A flag-staff 20 feet high stands on a wall 40 feet high. At a point E on a level with the bottom of the wall the flag-staff subtends an angle of 10°. Find the distance of E from the wall.

(10) On the bank of a river there is a column 200 feet high supporting a statue 30 feet high. The statue to an observer on the opposite bank subtends the same angle as a man 6 feet high standing at the base of the column. Find the breadth of the river.

210. Up to this point we have supplied the student with all the materials required for the solution of each example. But as he ought to have some practice in making extracts from the tables we shall suppose him to be in possession of a set of tables, and we shall now give a series of examples by which he may test his ability to apply the formulæ for the solution of Triangles.

Examples.—LIV.

(1) Given $a = 4$, $b = 3$, $C = 90^\circ$.

(2) Given $b = 55$, $c = 73$, $C = 90^\circ$.

(3) Given $a = 272$, $b = 225$, $C = 90^\circ$.

(4) Given $b = 399$, $c = 401$, $C = 90^\circ$.

(5) Given $c = 445$, $A = 10^\circ.\ 52'.\ 50''\cdot4$, $C = 90^\circ$.

(6) Given $c = 629$, $A = 46^\circ.\ 59'.\ 49''\cdot7$, $C = 90^\circ$.

(7) Given $c = 449$, $B = 51^\circ.\ 25'.\ 11''\cdot7$, $C = 90^\circ$.

(8) Given $c = 349$, $B = 58^\circ.\ 57'.\ 6''\cdot4$, $C = 90^\circ$.

(9) Given $a = 520$, $A = 66^\circ.\ 2'.\ 52''$, $C = 90^\circ$.

(10) Given $b = 31$, $A = 86^\circ.\ 18'.\ 17''$, $C = 90^\circ$.

Examples.—LV.

Solve the triangles, not right-angled, for which the following parts are given:

(1) $a = 197$, $b = 53$, $c = 240$.

(2) $a = 509$, $b = 221$, $c = 480$.

(3) $a = 533$, $b = 317$, $c = 510$.

(4) $a = 565$, $b = 445$, $c = 606$.

(5) $a = 409$, $b = 241$, $c = 182$.

(6) $b = 29$, $A = 43^\circ.\ 36'.\ 10''\cdot1$, $C = 124^\circ.\ 58'.\ 33''\cdot6$.

(7) $b = 149$, $A = 69^\circ.\ 59'.\ 2''\cdot5$, $C = 70^\circ.\ 42'.\ 30''$.

(8) $a = 101$, $b = 29$, $C = 32^\circ.\ 10'.\ 53''\cdot 8$.

(9) $a = 401$, $b = 41$, $C = 96^\circ.\ 57'.\ 20''\cdot 1$.

(10) $a = 221$, $b = 149$, $C = 30^\circ.\ 40'.\ 35''$.

(11) $a = 109$, $b = 61$, $C = 66^\circ.\ 59'.\ 25''\cdot 4$.

(12) $a = 445$, $b = 83$, $C = 87^\circ.\ 55'$.

(13) $a = 229$, $b = 109$, $C = 131^\circ.\ 24'.\ 44''$.

(14) $a = 241$, $b = 169$, $C = 104^\circ.\ 3'.\ 51''$.

(15) $a = 241$, $b = 169$, $C = 15^\circ.\ 22'.\ 37''$.

(16) $a = 13$, $b = 37$, $A = 18^\circ.\ 55'.\ 28''\cdot 7$, find B.

(17) $a = 445$, $b = 565$, $A = 44^\circ.\ 29'.\ 53''$, find B.

(18) $a = 212\cdot 5$, $b = 836\cdot 4$, $A = 14^\circ.\ 24'.\ 25''$, find B.

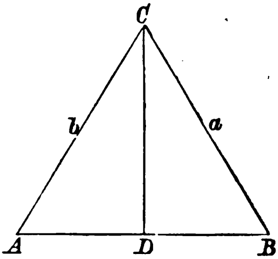
(19) $a = 379\cdot 5$, $b = 564\cdot 8$, $A = 40^\circ.\ 32'.\ 16''$, find B.

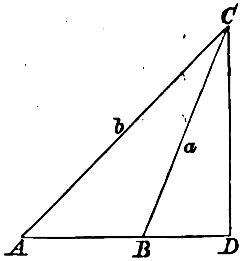
(20) $a = 9459\cdot 31$, $b = 8032\cdot 29$, $A = 71^\circ.\ 3'.\ 34''\cdot 7$, find B.

CHAPTER XVII.

PROPOSITIONS RELATING TO THE AREAS OF TRIANGLES, POLYGONS AND CIRCLES.

211. *Expressions for the area of a triangle.*





The area of a triangle is equal to half the rectangle contained by one of the sides and the perpendicular drawn to meet that side from the opposite angle.

Let ABC be the triangle, and as one of the angles A, B must be acute, let it be A. Draw a perpendicular from C to meet AB or AB produced in D.

$$\text{Then, area of triangle } ABC = \frac{1}{2} . AB . CD$$

$$= \frac{1}{2} . AB . AC . \sin A$$

$$= \frac{1}{2}\, cb. \sin A \, ;$$

that is, the area of a triangle is equal to *half the product of two sides and the sine of the angle between them.*

Also, since $\sin A = \frac{2}{bc} \cdot \sqrt{s \,.\, (s-a)(s-b)(s-c)}$, by Art. 181,

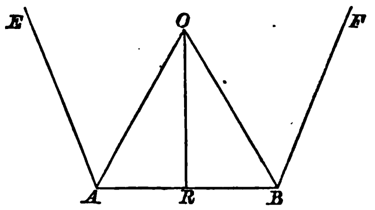
$$\text{area of triangle } ABC = \frac{1}{2} cb \,.\, \frac{2}{bc} \,.\, \sqrt{s \,.\, (s-a)(s-b)(s-c)}$$

$$= \sqrt{s \,.\, (s-a)(s-b)(s-c)},$$

which gives an expression for the area in terms of the sides.

For this expression the symbol used is S.

212. *To find the area of a regular polygon in terms of its side.*



Let EA, AB, BF be three consecutive sides of a regular polygon of n sides and let each of them $= a$.

Bisect the angles EAB, ABF by the lines OA, OB meeting in O.

Draw OR at right angles to AB.

Now angle $AOR = \frac{AOB}{2}$,

and angle $AOB = \frac{2\pi}{n}$ (Eucl. I. 15, Cor.);

$$\therefore \text{ angle } AOR = \frac{\pi}{n};$$

Hence area of polygon $= n$ times area of triangle AOB

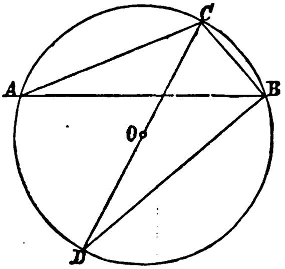
$$= n \, . \, AR \, . \, RO$$

$$= n \, . \, AR \, . \, AR \, . \cot AOR$$

$$= n \, . \, \frac{a}{2} \, . \, \frac{a}{2} \, . \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \, . \cot \frac{\pi}{n} \, .$$

213. *To find the radius of a circle described about a triangle in terms of the sides of the triangle.*



Let O be the centre of the circle described about the triangle ABC, and R its radius.

Through O draw the diameter CD and join BD.

Then CBD, being the angle in a semicircle, is a right angle.

And BDC = angle CAB in the same segment $= A$.

Now $\frac{CB}{CD} = \sin BDC$,

that is, $\frac{a}{2R} = \sin A$;

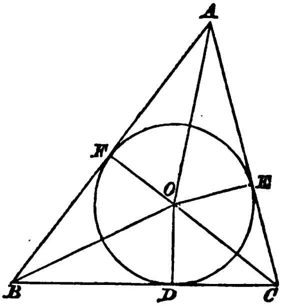
$$\therefore a = 2R \, . \sin A;$$

$$\therefore R = \frac{a}{2 \sin A} \, .$$

But, by Art. 181, $\sin A = \frac{2}{bc} . S$;

$$\therefore R = \frac{abc}{4S}.$$

214. *To find the radius of a circle inscribed in a triangle in terms of the sides of the triangle.*



Let O be the centre of the inscribed circle, and r its radius.

Then S = area of ABC

$$= \text{area of } BOC + \text{area of } COA + \text{area of } AOB$$

$$= \frac{OD . BC}{2} + \frac{OE . AC}{2} + \frac{OF . AB}{2}$$

$$= \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$$

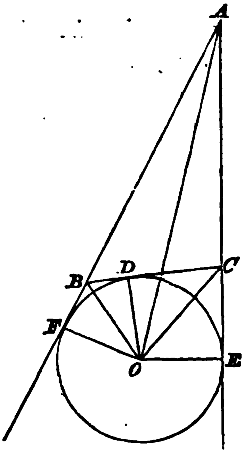
$$= r . \frac{a+b+c}{2}$$

$$= rs;$$

$$\therefore r = \frac{S}{s}.$$

215. *To find the radii of the circles escribed, that is which touch one of the sides of a triangle and the other sides produced.*

Let O be the centre of the escribed circle that touches the side BC and the other sides produced, and let the radius of this circle be r_1.



Then quadrilateral $ABOC$ = triangle AOC + triangle AOB

and quadrilateral $ABOC$ = triangle ABC + triangle BOC;

$$\therefore AOC + AOB = ABC + BOC,$$

$$\therefore \frac{AC\,.\,OE}{2} + \frac{AB\,.\,OF}{2} = S + \frac{BC\,.\,OD}{2},$$

$$\therefore \frac{br_1}{2} + \frac{cr_1}{2} = S + \frac{ar_1}{2},$$

$$\therefore \frac{b+c-a}{2}\,.\,r_1 = S,$$

$$\therefore (s-a)\,r_1 = S,$$

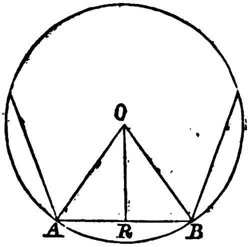
$$\therefore r_1 = \frac{S}{s-a};$$

Similarly it may be shewn that if r_2, r_3 are the radii of the circles touching AC and AB respectively,

$$r_2 = \frac{S}{s-b},$$

$$r_3 = \frac{S}{s-c}.$$

216. *To find the area of a regular polygon inscribed in a circle.*



Let O be the centre of the circle, r the radius of the circle, AB a side of the polygon.

Join OA, OB.

Then area of polygon $= n$ times area of triangle AOB

$$= n \cdot \frac{1}{2} AO \cdot OB \cdot \sin AOB, \text{ (Art. 211)}$$

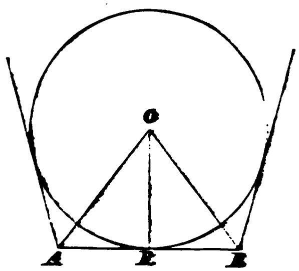
$$= n \cdot \frac{1}{2} \cdot r \cdot r \cdot \sin \frac{2\pi}{n}$$

$$= \frac{nr^2}{2} \cdot \sin \frac{2\pi}{n}.$$

217. *To find the area of a regular polygon described about a circle.*

Let O be the centre of the circle, r the radius, AB a side of the circumscribing polygon.

Draw the radius OR at right angles to AB.



Then area of polygon = n times area of triangle AOB

$$= n \, . \, OR \, . \, AR$$

$$= n \, . \, OR \, . \, OR \tan AOR$$

$$= n \, . \, r \, . \, r \, . \tan \frac{\pi}{n}$$

$$= nr^2 \, . \tan \frac{\pi}{n} \, .$$

218. *To find the area of a circle.*

Taking the figure and notation of the preceding Article,

area of circumscribing polygon = n times area of triangle AOB

$$= n \, . \, \frac{1}{2} \, . \, AB \, . \, OR$$

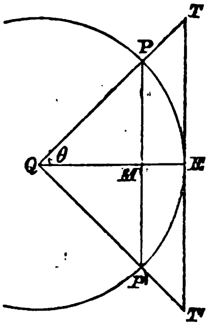
$$= \frac{1}{2} OR \times n \, . \, AB$$

$$= \frac{1}{2} OR \times \text{perimeter of polygon.}$$

Now if the number of sides of the polygon be indefinitely increased and the length of each side indefinitely diminished, the perimeter of the polygon coincides with the circumference of the circle, and the area of the polygon is the same as the area of the circle;

$$\therefore \text{ area of circle} = \frac{1}{2} OR \times \text{circumference of circle}$$
$$= \frac{1}{2} r \times 2\pi r$$
$$= \pi r^2.$$

219. *To shew that if θ be the circular measure of a positive angle less than a right angle, $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude.*



Let Q be the centre of a circle, QE a radius cutting the chord PP' at right angles, TT' a tangent to the circle at E.

Let the circular measure of the angle EQP be θ.

Then
$$\sin\theta = \frac{PM}{QP},$$
$$\theta = \frac{PE}{QP},$$
$$\tan\theta = \frac{TE}{QE} = \frac{TE}{QP}.$$

Hence assuming that PE is greater than PM but less than TE, PM, PE, TE are in ascending order of magnitude.

Therefore $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

220. *To shew that when θ is indefinitely diminished,*

$$\frac{\theta}{\sin\theta}=1.$$

Since $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude,

$\sin\theta$, θ, $\frac{\sin\theta}{\cos\theta}$ are in ascending order of magnitude.

Divide each by $\sin\theta$, then

1, $\frac{\theta}{\sin\theta}$, $\frac{1}{\cos\theta}$ are in ascending order of magnitude;

$\therefore$ $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$.

Now when $\theta=0$, $\cos\theta=1$,

and therefore $\frac{1}{\cos\theta}=1$;

therefore when $\theta=0$,

$$\frac{\theta}{\sin\theta}=1.$$

EXAMPLES.—LVI.

1. The angle included between two sides of a triangle whose lengths are 10 inches and 12 inches is 60°: find the area of the triangle.

2. Two sides of a triangle are 40 and 60 feet, and they contain an angle of 30°: find the area of the triangle.

3. What is the area of a triangle whose base is 4 feet, and altitude $1\frac{1}{4}$ yards?

4. What is the area of a triangle whose sides are 5, 6, 5 inches respectively?

5. If $a = 625$, $b = 505$, $c = 904$, what is the measure of the area of the triangle?

6. If $a = 409$, $b = 169$, $c = 510$, what is the measure of the area of the triangle?

7. If $a = 577$, $b = 73$, $c = 520$, what is the measure of the area of the triangle?

8. In a right-angled triangle area $= s.(s - c)$, C being the right angle.

9. If $a = 52{\cdot}53$, $b = 48{\cdot}76$, $c = 44{\cdot}98$, $\log 146{\cdot}27 = 2{\cdot}1651553$,
$\log 56{\cdot}31 = 1{\cdot}7505855$, $\log 48{\cdot}75 = 1{\cdot}6879746$,
$\log 41{\cdot}21 = 1{\cdot}6150026$, $\log 2 = {\cdot}3010300$,
$\log 1{\cdot}0169487 = {\cdot}0072990$, find the measure of area.

10. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Shew that its largest angle is 60°.

11. In the rectangular sheet of paper $ABCD$, the angular point A is turned down so as to lie in the side CD, while the crease of the paper passes through the angular point B; shew that the area of the part turned down is

$$\frac{1}{2} BC\{AB - \sqrt{(AB^2 - BC^2)}\}.$$

12. Shew that the area of a triangle

$$= \frac{a^2 . \sin B . \sin C}{2 \sin (B + C)}.$$

13. In any triangle the area

$$= \sin \frac{A}{2} . \sin \frac{B}{2} . \sin \frac{C}{2} \left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C}\right).$$

14. If the radius of the inscribed circle be equal to half that of the circumscribed circle, the triangle is equilateral.

15. In any triangle

$$(b-c)\cos\frac{A}{2}=a\,.\,\sin\frac{B-C}{2}\,.$$

16. If the points of contact of a triangle inscribed in a circle with the sides be joined, shew that the area of the triangle so formed

$$=\frac{2}{abc}\,s^{\frac{1}{2}}\{(s-a)\,(s-b)\,(s-c)\}^{\frac{3}{2}}.$$

17. The diagonals of a quadrilateral are in length a, b respectively, and intersect at an angle A. Shew that its area

$$=\frac{1}{2}\,ab\sin A.$$

18. The area of any triangle

$$=\frac{a^2-b^2}{2}\,.\,\frac{\sin A\,.\,\sin B}{\sin(A-B)}\,.$$

19. In an isosceles right-angled triangle, shew that the radius of one of the equal escribed circles is equal to the radius of the circumscribed circle.

20. In any right-angled triangle, C being the right angle,

$$\cot(B-A)+\cot 2\left(A+\frac{C}{2}\right)=0.$$

21. The area of any triangle

$$=\frac{2abc}{a+b+c}\,.\cos\frac{A}{2}\,.\cos\frac{B}{2}\,.\cos\frac{C}{2}\,.$$

22. In any isosceles triangle, C being the vertical angle,

$$\text{area}\times 32\cos^4\frac{A}{2}=\sin 2A\,(2a+c)^2.$$

23. The length of a perpendicular from A to BC

$$=\frac{b^2\sin C+c^2\sin B}{b+c}\,.$$

24. Taking the notation adopted in this Chapter, prove the following relations:

(1) $$r=\frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}}.$$

(2) $$r=\frac{2R.\sin A.\sin\frac{1}{2}B.\sin\frac{1}{2}C}{\cos\frac{1}{2}A}.$$

(3) $$r_1=\frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}.$$

(4) $$r_1=4R.\sin\frac{A}{2}\cos\frac{B}{2}.\cos\frac{C}{2}.$$

(5) $$r_1+r_2+r_3=R(3+\cos A+\cos B+\cos C).$$

(6) $$R+r=R(\cos A+\cos B+\cos C).$$

25. Shew that the area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

26. The distances between the centre of the inscribed and those of the escribed circles of a triangle ABC are

$$4R.\sin\frac{A}{2},\quad 4R\sin\frac{B}{2},\quad 4R\sin\frac{C}{2},$$

R being the radius of the circumscribing circle.

27. The points at which the lines bisecting the angles A, B, C of a triangle cut the opposite sides are joined. Shew that the area of the triangle so formed bears to that of the triangle ABC the ratio

$$\frac{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B-C}{2}.\cos\frac{C-A}{2}.\cos\frac{A-B}{2}}.$$

28. If r_1, r_2, r_3 be the radii of the escribed circles and s the semi-perimeter of the triangle, shew that

$$s^2 = r_1r_2 + r_2r_3 + r_3r_1.$$

29. Prove that the lines drawn from the centres of the inscribed to those of the escribed circles are each equal to twice the radius of the circumscribed circle.

30. Shew that the distances from the centre of the inscribed circle to the centres of the escribed circles are respectively equal to

$$\frac{a}{\cos\frac{A}{2}},\ \frac{b}{\cos\frac{B}{2}},\ \frac{c}{\cos\frac{C}{2}}.$$

31. r is the radius of a circle inscribed in a triangle ABC; shew that

$$a = r \operatorname{cosec}\frac{A}{2} \operatorname{cosec}\frac{B}{2} \operatorname{cosec}\frac{C}{2}.$$

32. If R, r be the radii of the circles described about and inscribed in the triangle ABC, and s the semi-perimeter of the triangle, prove that

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \frac{s}{r} = 4R\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

ANSWERS.

I. (Page 2.)

(1) 54. (2) 26. (3) 4 inches. (4) 5 inches.

(5) 3 inches. (6) 5 inches. (7) $\frac{b}{3a}$. (8) $\frac{cn}{3m}$.

II. (Page 3.)

(1) 45 yds. (2) 10 ft. (3) 255 yds. (4) 360·5...yds.

(5) 163·25 yds. nearly. (6) 12 ft., 16 ft. (8) 42 ft., 30 ft.

(9) $6\sqrt{2}$ ft. (10) $5\sqrt{2}$ inches. (11) $625\sqrt{2}$ ft.

(12) $13\sqrt{3}$. (13) $10\sqrt{3}$. (14) 12 inches.

(15) 38 inches. (16) 634 ft.

III. (Page 9.)

(1) $15\frac{5}{7}$ ft. (2) $86{\cdot}352\dot{2}\dot{7}$ ft. (3) $25{\cdot}\dot{7}1428\dot{5}$ miles.

(4) $7954\frac{6}{11}$ miles. (5) $2775834\frac{2}{7}$ miles. (6) $1089\frac{17}{22}$ miles.

(7) 6 ft. $6\frac{4}{7}$ in. (8) $47\frac{1}{7}$ ft. (9) $\frac{525\sqrt{2}}{22}$ ft.

(10) $\frac{350\sqrt{2}}{11}$ ft. (11) $12\frac{6}{7}$ miles. (12) $22\frac{1}{2}$ miles.

IV. (Page 14.)

(1) $24\cdot2680\dot{5}$. (2) $37\cdot0452\dot{7}$. (3) $175\cdot003\dot{8}$.

(4) $\cdot09\dot{1}$. (5) $375\cdot0\dot{6}$. (6) $78\cdot20\dot{2}$.

V. (Page 15.)

(1) 25·1425. (2) 38·0415. (3) 214·0307.

(4) ·150745. (5) 425·130554. (6) 2·020222.

VI. (Page 20.)

(1) $30^{g}.\ 29^{\backprime}.\ 19^{\backprime\backprime}\cdot75\ldots$ (2) $174^{g}.\ 52^{\backprime}.\ 12^{\backprime\backprime}\cdot96\ldots$ (3) $45^{\backprime}$.

(4) $21^{g}.\ 11^{\backprime}.\ 66^{\backprime\backprime}\cdot\dot{6}$. (5) $159^{g}.\ 5^{\backprime}.\ 55^{\backprime\backprime}\cdot\dot{5}$. (6) $31^{g}.\ 11^{\backprime}.\ 11^{\backprime\backprime}\cdot\dot{1}$.

(7) $11^{g}.\ 58^{\backprime}.\ 89^{\backprime\backprime}\cdot3$. (8) $30^{g}.\ 70^{\backprime}.\ 74^{\backprime\backprime}\cdot1$. (9) $333^{g}.\ 62^{\backprime}.\ 90^{\backprime\backprime}\cdot1$.

(10) $469^{g}.\ 2^{\backprime}.\ 53^{\backprime\backprime}\cdot3\ldots$

VII. (Page 21.)

(1) $17^{\circ}.\ 30'.\ 48''\cdot78$. (2) $111^{\circ}.\ 38'.\ 44''\cdot592$.

(3) $26^{\circ}.\ 46'.\ 30''$. (4) $13^{\circ}.\ 30'.\ 4''\cdot86$. (5) $138^{\circ}.\ 39'.\ 54''\cdot576$.

(6) $38^{\circ}.\ 42'$. (7) $34^{\circ}.\ 50'.\ 27''$. (8) $45^{\circ}.\ 41'.\ 33''$.

(9) $153^{\circ}.\ 34'.\ 16''$. (10) $291^{\circ}.\ 43'.\ 30''$.

VIII. (Page 22.)

(1) $\frac{\pi}{3}$. (2) $\frac{\pi}{8}$. (3) $\frac{\pi}{16}$. (4) $\frac{3\pi}{2}$.

(5) $\frac{7\pi}{4}$. (6) $\frac{1453\pi}{10800}$. (7) $\frac{143\pi}{270}$. (8) $\frac{2719\pi}{40500}$.

(9) $\frac{\pi}{3}$ each. (10) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$.

IX. (Page 22.)

(1) 90°. (2) 60°. (3) 45°. (4) 30°. (5) 120°.

(6) $\frac{90}{\pi}$ degrees. (7) $\frac{60}{\pi}$ degrees. (8) $\frac{45}{\pi}$ degrees.

(9) $\frac{30}{\pi}$ degrees. (10) $\frac{120}{\pi}$ degrees.

X. (Page 23.)

(1) $\frac{\pi}{4}$. (2) $\frac{\pi}{8}$. (3) $\frac{\pi}{32}$. (4) $\frac{5\pi}{4}$. (5) $\frac{5\pi}{2}$.

(6) $\cdot 0652525\pi$. (7) $\cdot 120751075\pi$. (8) $\cdot 6250065\pi$.

(9) $\cdot 00015\pi$. (10) $\cdot 0000025\pi$.

XI. (Page 23.)

(1) $66\cdot\dot{6}$ grades. (2) 40^g. (3) $33\cdot\dot{3}$ grades.

(4) $133\cdot\dot{3}$ grades. (5) 120^g. (6) $\frac{200}{3\pi}$ grades.

(7) $\frac{40}{\pi}$ grades. (8) $\frac{25}{\pi}$ grades. (9) $\frac{120}{\pi}$ grades.

(10) $\frac{460}{\pi}$ grades.

XII. (Page 23.)

(1) $4\cdot 5$. (2) $4\cdot 25$ degrees. (3) $3\frac{1}{5}^\circ$, $\frac{5}{8}$, $\frac{8}{5}$.

(4) $3\frac{1}{2}^\circ$, $\frac{6}{7}$, $\frac{7}{6}$. (5) $\frac{14}{15}$. (6) $70 : 67$.

(8) 90°, 60°, 30°; 100^g, $66\frac{2}{3}^g$, $33\frac{1}{3}^g$. (9) $3\cdot 14159$.

(10) 36°, 54°, 90°; 40^g, 60^g, 100^g; $\frac{\pi}{5}$, $\frac{3\pi}{10}$, $\frac{\pi}{2}$.

(11) $173\frac{1}{3}^\circ$. (12) $125\cdot\dot{9}2\dot{5}^g$. (14) $\frac{800}{3\pi}$ grades.

(15) 700 : 11. (16) $\frac{1}{180}$th of a degree. (17) ·1775.

(18) $35^\circ.\,6'.\,38''\cdot88$. (19) 20π degrees. (20) $\frac{1}{36000}$th part.

(21) $\frac{1}{360}, \frac{1}{400}, \frac{1}{2\pi}$. (22) $\frac{5000m}{27}$. (23) $63\frac{7}{11}$ grades.

(24) $\frac{1}{81}$. (25) $67\frac{1}{2}^\circ$, 180°, $(n.180+15)^\circ$; $\frac{3\pi}{8}$, π, $n\pi+\frac{\pi}{12}$.

(26) $1\cdot9^\circ$. (27) $\frac{\pi}{12}$. (29) 120°, 108°. (30) 120^g, 150^g.

(31) $\frac{\pi}{3}, \frac{2\pi}{3}$. (33) $\pi-\frac{2\pi}{n}$. (34) π feet.

(35) 20°, 60°, 100°. (36) $\frac{450^\circ}{\pi}$. (37) $\frac{1600\pi-3600}{9\pi}$ grades.

(38) $54^\circ.\,46'.\,54''\cdot5$. (39) $\frac{133\pi}{6}$ miles. (40) 8 and 4.

XIII. (Page 36.)

(1) $\frac{BD}{AB}, \frac{AD}{AB}, \frac{BD}{AD}; \frac{AD}{AB}, \frac{BD}{AD}, \frac{AB}{AD}; \frac{BD}{BC}, \frac{CD}{BC}, \frac{DB}{DC}$.

(2) $\frac{OR}{OP}, \frac{PR}{OP}, \frac{PR}{OR}; \frac{OR}{OQ}, \frac{QR}{OQ}, \frac{OR}{QR}; \frac{QO}{OR}, \frac{RO}{QR}, \frac{QO}{QR}$.

XIV. (Page 47.)

(1) $\frac{1}{2\sqrt{2}}$. (2) $\sqrt{\frac{2}{3}}$. (3) $\frac{\sqrt{3}}{2}$. (4) $\sqrt{3}$. (5) $3\sqrt{2}+\sqrt{3}$.

XV. (Page 50.)

(1) 346·4 feet. (2) 247·48 feet. (3) 60°. (4) 173·2 feet.
(5) 464·34 feet. (6) 51·3 feet. (7) 42·25 ft. above the tower.

(8) 72 feet. (9) 150 feet. (10) 25°. (11) 139·5 feet.

(12) 125 yards. (13) $92\frac{4}{13}$ feet. (14) 42 feet. (15) 8·053 feet.

XVII. (Page 58.)

(1) $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \dfrac{\sqrt{1-\cos^2 A}}{\cos A}$, $\sec A = \dfrac{1}{\cos A}$,

$$\operatorname{cosec} A = \frac{1}{\sqrt{1-\cos^2 A}}, \quad \cot A = \frac{\cos A}{\sqrt{1-\cos^2 A}}.$$

(2) $\sin A = \dfrac{1}{\operatorname{cosec} A}$, $\cos A = \dfrac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$, $\tan A = \dfrac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$,

$$\sec A = \frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}, \quad \cot A = \sqrt{\operatorname{cosec}^2 A - 1}.$$

(3) $\sin A = \dfrac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \dfrac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$,

$$\operatorname{cosec} A = \frac{\sec A}{\sqrt{\sec^2 A - 1}}, \quad \cot A = \frac{1}{\sqrt{\sec^2 A - 1}}.$$

(4) $\sin A = \dfrac{1}{\sqrt{1+\cot^2 A}}$, $\cos A = \dfrac{\cot A}{\sqrt{1+\cot^2 A}}$, $\tan A = \dfrac{1}{\cot A}$,

$$\operatorname{cosec} A = \sqrt{1+\cot^2 A}, \quad \sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

XVIII. (Page 59.)

(1) $\dfrac{\sqrt{5}}{3}$, $\dfrac{2}{\sqrt{5}}$. (2) $\dfrac{3}{5}$, $\dfrac{3}{4}$. (3) $\dfrac{\sqrt{7}}{4}$, $\dfrac{3}{\sqrt{7}}$.

(4) $\sqrt{\dfrac{2}{3}}$, $\dfrac{1}{\sqrt{2}}$. (5) $\dfrac{\sqrt{(a^4+b^4)}}{a^2}$, $\dfrac{\sqrt{(a^4+b^4)}}{b^2}$.

(6) $\dfrac{\sqrt{(b^2-a^2)}}{a}$, $\dfrac{b}{\sqrt{(b^2-a^2)}}$. (7) $\dfrac{a}{\sqrt{(1-a^2)}}$, $\dfrac{1}{\sqrt{(1-a^2)}}$.

(8) $\frac{\sqrt{(1-b^2)}}{b}$, $\frac{1}{\sqrt{(1-b^2)}}$. (9) $\frac{4}{5}$, $\frac{4}{3}$.

(10) $\frac{5}{2\sqrt{14}}$, $\frac{9}{2\sqrt{14}}$. (11) $\frac{\sqrt{403}}{22}$, $\frac{\sqrt{403}}{9}$.

(12) $\frac{\sqrt{549}}{93}$, $\frac{\sqrt{549}}{90}$, (13) $\frac{20}{101}$, $\frac{20}{99}$.

(14) $\frac{99}{101}$, $\frac{99}{20}$. (15) $\frac{5}{13}$, $\frac{13}{12}$.

In Ex. XIX. (13) for $\sin^2 b$ read $\sec^2 b$.

XX. (Page 63.)

1. (1) $65^\circ.45'.18''$. (2) $46^\circ.57'.3''$. (3) $25^\circ.59'.46''$.
(4) $7^\circ.55'.45''$. (5) $-(35^\circ.15'.42'')$. (6) $-(88^\circ.27'.34'')$.
(7) -105°. (8) -164°. (9) 115°. (10) 335°.

2. (1) $67^g.76^{`}.76^{``}$. (2) $4^g.96^{`}.25^{``}$. (3) $53^g.99^{`}.16^{``}$.
(4) $97^g.94^{`}.96^{``}$. (5) $-(35^g.2^{`}.5^{``})$. (6) $-(69^g.0^{`}.3^{``})$.
(7) -143^g. (8) -257^g. (9) 135^g. (10) 345^g.

3. (1) $\frac{\pi}{4}$. (2) $\frac{\pi}{6}$. (3) $-\frac{\pi}{10}$. (4) $\frac{3\pi}{4}$. (5) $\frac{5\pi}{4}$.

XXI. (Page 66.)

1. (1) $145^\circ.47'.11''$. (2) $47^\circ.35'.13''$. (3) $33^\circ.59'.19''$.
(4) $151^\circ.44'.56''$. (5) $1''$. (6) $79^\circ.10'.7''$.
(7) -65°. (8) $-(257^\circ.3'.4'')$. (9) 229°. (10) 535°.

2. (1) $67^g.67^{`}.58^{``}$. (2) $4^g.97^{`}.43^{``}$. (3) $196^g.2^{`}.2^{``}$.
(4) $134^g.87^{`}.92^{``}$. (5) $45^g.96^{`}.94^{``}$. (6) $25^g.99^{`}.96^{``}$.
(7) -75^g. (8) $-(327^g.2^{`}.14^{``})$. (9) 235^g. (10) 525^g.

3. (1) $\frac{\pi}{2}$. (2) $\frac{2\pi}{3}$. (3) $\frac{\pi}{5}$. (4) $\frac{5\pi}{4}$. (5) $\frac{7\pi}{4}$.

4. π.

XXIII. (Page 70.)

(1) $\frac{\sqrt{3}}{2}$. (2) $-\frac{1}{2}$. (3) $\frac{1}{\sqrt{2}}$. (4) $-\frac{1}{\sqrt{2}}$.

(5) $\frac{1}{2}$. (6) $-\frac{\sqrt{3}}{2}$. (7) $-\frac{1}{\sqrt{2}}$. (8) $-\frac{\sqrt{3}}{2}$.

(9) $-\sqrt{3}$. (10) $-\frac{2}{\sqrt{3}}$. (11) $\sqrt{2}$. (12) $-\sqrt{3}$.

XXIV. (Page 73.)

(1) -45°. (2) 45°. (3) 0°. (4) 90°.

(5) 0° or 60°. (6) 30°. (7) 30°. (8) 45°.

(9) 30°. (10) 60°. (11) 30° or 60°. (12) 30° or 60°.

(13) 45°. (14) 45°. (15) 45°. (16) 45°.

(17) 135°. (18) 45°. (19) 30° or 90°. (20) 45°.

XXV. (Page 79.)

(1) $\frac{\sqrt{3}}{2}$. (2) $-\frac{1}{2}$. (3) $\frac{1}{\sqrt{2}}$. (4) $-\frac{1}{\sqrt{2}}$.

(5) $\frac{1}{2}$. (6) $-\frac{\sqrt{3}}{2}$. (7) $-\frac{1}{\sqrt{2}}$. (8) $-\frac{\sqrt{3}}{2}$.

(9) $-\sqrt{3}$. (10) $-\frac{2}{\sqrt{3}}$. (11) $\sqrt{2}$. (12) $-\sqrt{3}$.

(13) $\frac{1}{2}$. (14) $\frac{2}{\sqrt{3}}$. (15) 1. (16) $\frac{\sqrt{3}}{2}$.

(17) $\frac{1}{\sqrt{2}}$. (18) $\frac{\sqrt{3}}{2}$. (19) 0. (20) 1.

(21) -2. (22) $\frac{1}{\sqrt{3}}$. (23) 1. (24) $-\frac{2}{\sqrt{3}}$.

(25) $\frac{\sqrt{3}}{2}$. (26) 1. (27) $-\sqrt{2}$. (28) -1.

XXVI. (Page 80.)

(1) $n\pi + (-1)^n . \frac{\pi}{2}$. (2) $2n\pi$. (3) $n\pi + (-1)^n . \frac{\pi}{4}$.

(4) $n\pi + \frac{\pi}{3}$. (5) $n\pi + (-1)^n \frac{\pi}{6}$. (6) $n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

(7) $n\pi + (-1)^n . \frac{\pi}{4}$. (8) $2n\pi \pm \frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$.

(9) $n\pi + \frac{\pi}{4}$. (10) $2n\pi \pm \frac{\pi}{3}$.

In Ex. XXVII. for $\sin^2\alpha - \sin^2\beta$ read $\sin^2\beta - \sin^2\alpha$.

XXVIII. (Page 86.)

(4) $\frac{\sqrt{5} + 2\sqrt{8}}{9}$. (5) $\frac{8\sqrt{7} - 9\sqrt{21}}{60}$.

(6) $\frac{\sqrt{3} - 1}{2\sqrt{2}}$. (7) $\frac{\sqrt{3} + \sqrt{899}}{60}$.

XXX. (Page 87.)

(1) 45°. (2) 0°. (3) 105°. (4) 30°. (5) 45°.
(6) 60°.

XXXII. (Page 91.)

(1) $2 \cos \frac{1}{2}\left(\alpha + \frac{\pi}{2} - \beta\right) \sin \frac{1}{2}\left(\alpha - \frac{\pi}{2} + \beta\right)$.

(2) $2 \sin \frac{1}{2}(\pi + \alpha - \theta) . \cos(\alpha)$.

(3) $2 \sin \frac{\pi}{4} . \cos\left(\alpha - \frac{\pi}{4}\right)$.

(4) $2 \cos \frac{\pi}{4} \sin\left(\alpha - \frac{\pi}{4}\right)$. (5) $2 \sin 25° . \cos 5°$.

(6) $2 \cos 15° . \sin 5°$. (7) $2 \sin \frac{7\pi}{24} \cos \frac{\pi}{24}$.

(8) $2 \cos \frac{19\pi}{60} . \sin \frac{\pi}{60}$.

XXXV. (Page 103.)

2.

(1) $\theta = 45^\circ$ or -15°. (2) $\theta = 30^\circ$. (3) $x = 0^\circ$ or $7\frac{1}{2}^\circ$.

(4) $\theta = 15^\circ$ or 30°. (5) $A = 30^\circ$. (6) $\theta = 15^\circ$.

(7) $\theta = 18^\circ$ or 54°. (8) $\alpha = 0^\circ, 30^\circ$. (9) $\theta = 18^\circ, 234^\circ$.

(10) $\alpha = 15^\circ, 60^\circ$. (11) $\theta = \frac{\pi}{6}$. (12) $\theta = 0^\circ$ or 30°.

XXXVI. (Page 104.)

(1) $\frac{1}{4}\sqrt{(10 - 2\sqrt{5})}$. (2) $\frac{1}{4}(1 + \sqrt{5})$. (3) $\frac{1}{4}(1 + \sqrt{5})$.

(4) $\frac{1}{4}\sqrt{(10 - 2\sqrt{5})}$. (5) $\frac{1}{4}\sqrt{(10 + 2\sqrt{5})}$.

(6) $\frac{\sqrt{(10 + 2\sqrt{5})}}{\sqrt{5} - 1}$. (7) 1. (8) 0.

XXXVII. (Page 108.)

(1) $\sin\frac{A}{2} + \cos\frac{A}{2} = +\sqrt{1 + \sin A}$;

$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1 - \sin A}$.

(2) $\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1 + \sin A}$;

$\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1 - \sin A}$.

(3) $\cos 189^\circ = -\frac{1}{4}\left\{\sqrt{5 - \sqrt{5}} + \sqrt{3 + \sqrt{5}}\right\}$;

$\sin 189^\circ = \frac{1}{4}\left\{\sqrt{5 - \sqrt{5}} - \sqrt{3 + \sqrt{5}}\right\}$.

(4) $\frac{2 - \sqrt{2}}{2\sqrt{3}}$. (5) $-\sqrt{\left(\frac{2 + \sqrt{2}}{2}\right)}$.

XXXIX. (Page 118.)

(1) $\bar{1}\cdot2187180$. (2) $\bar{7}\cdot7074922$. (3) $2\cdot4036784$.
(4) $4\cdot740378$. (5) $2\cdot824059$. (6) $\bar{3}\cdot724833$.
(7) $\bar{5}\cdot3790163$. (8) $\overline{40}\cdot578098$. (9) $\overline{62}\cdot9905319$.
(10) $\bar{2}\cdot1241803$. (11) $\bar{3}\cdot738827$. (12) $\bar{1}\cdot61514132$.

XL. (Page 121.)

(1) 2·1072100; 2·0969100; 3·3979400.
(2) 1·6989700; $-2\cdot3010300$; 2·2922560.
(3) ·7781513; 1·4313639; 1·7323939; 2·7604226.
(4) 1·7781513; $-1\cdot5228787$; ·0211893; $-4\cdot3645161$.
(5) $-3\cdot1249387$; 1·4983106.
(6) ·3010300; $-1\cdot1938200$; ·2916000.
(7) ·6989700; $-\cdot9030900$; 3·391073.
(8) $-2, 0, 2 : 1, 0, -1$.
(9) (1) 3. (2) 2. (10) $x=\frac{9}{2}$, $y=\frac{3}{2}$.
(11) (*a*) ·3010300; 1·3979400; 1·9201233.
(*b*) 103. (*c*) 17·6 ... years.
(12) (*a*) ·6989700; ·6020600; 1·7118072; $-\cdot0119382$.
(*b*) 8. (*c*) 23·4 ... years.
(13) 3·8821260; $-1\cdot5906306$; $-2\cdot2544674$.
(14) (1) $x=\frac{1}{6}$. (2) $x=2$. (3) $x=\dfrac{\log m}{\log a+\log b}$.

(4) $x=\dfrac{\log c}{m\log a+2\log b}$. (5) $x=\dfrac{4\log b+\log c}{2\log c+\log b-3\log a}$.

(6) $x=\dfrac{\log c}{\log a+m\log b+3\log c}$.

XLI. (Page 126.)

(1) 4·7201798. (2) 2·477736. (3) 1·5054974.
(4) 2·3740194. (5) 1·8293175. (6) $\bar{3}$·8653133.
(7) $\bar{6}$·8190944. (8) 3·5324716. (9) 5·5921479.
(10) ·4119438.

XLII. (Page 128.)

(1) 12954·8. (2) 4624·51. (3) 345·7291.
(4) 393756·9. (5) 3715·9523. (6) ·109646152.
(7) ·00000025725987. (8) 601·9541.
(9) 1090·529. (10) 262·0188.

XLIII. (Page 131.)

(1) ·672367. (2) ·9523071. (3) ·8151278.
(4) ·999000. (5) ·6850417. (6) ·752140.
(7) ·024003. (8) 1·399364. (9) ·9230769.
(10) ·82$\dot{7}$0$\dot{2}$.

XLIV. (Page 134.)

(1) 48°. 46′. 34″. (2) 2°. 33′. 45″. (3) 43°. 14′.
(4) 32°. 31′. 13″·5. (5) 24°. 11′. 22″·3. (6) 82°. 22′. 18″·7.
(7) 53°. 7′. 48″·4. (8) 25°. 3′. 27″. (9) 73°. 44′. 23″·3.
(10) 77°. 19′. 20″.

XLV. (Page 137.)

(1) 9·9163315. (2) 9·6912284. (3) 9·8996023.
(4) 9·9091760. (5) 9·7203472. (6) 8·9610073.
(7) 11·1975677. (8) 9·8027682. (9) 9·9745378.
(10) 8·281485.

XLVI. (Page 139.)

(1) $14^\circ. 24'. 35''$. (2) $54^\circ. 13'. 19''$. (3) $71^\circ. 40'. 18''$.
(4) $29^\circ. 25'. 2''$. (5) $30^\circ. 50'. 27''\cdot 6$. (6) $86^\circ. 32'. 24''\cdot 5$.
(7) $24^\circ. 8'. 45''$. (8) $11^\circ. 39'. 52''$. (9) $46^\circ. 23'. 11''$.
(10) $29^\circ. 54'. 29''\cdot 7$.

XLIX. (Page 156.)

(1) $a = 4$, $A = 53^\circ. 7'. 48''\cdot 4$, $B = 36^\circ. 52'. 11''\cdot 6$.
(2) $a = 8$, $A = 28^\circ. 4'. 20''\cdot 9$, $B = 61^\circ. 55'. 39''\cdot 1$.
(3) $a = 20$, $A = 43^\circ. 36'. 10''\cdot 1$, $B = 46^\circ. 23'. 49''\cdot 9$.
(4) $a = 24$, $A = 73^\circ. 44'. 23''\cdot 3$, $B = 16^\circ. 15'. 36''\cdot 7$.
(5) $a = 56$, $A = 59^\circ. 29'. 23''\cdot 2$, $B = 30^\circ. 30'. 36''\cdot 8$.
(6) $a = 12$, $b = 5$, $B = 22^\circ. 37'. 11''\cdot 5$.
(7) $a = 40$, $b = 9$, $B = 12^\circ. 40'. 49''\cdot 4$.
(8) $a = 48$, $b = 55$, $A = 41^\circ. 6'. 43''\cdot 5$.
(9) $a = 80$, $b = 39$, $A = 25^\circ. 59'. 21''\cdot 2$.
(10) $b = 9$, $c = 41$, $B = 12^\circ. 40'. 49''\cdot 4$.

L. (Page 158.)

(1) $b = 153$, $A = 34^\circ. 12'. 19''\cdot 6$, $B = 55^\circ. 47'. 40''\cdot 4$.
(2) $b = 297$, $A = 45^\circ. 40'. 2''\cdot 3$, $B = 44^\circ. 19'. 57''\cdot 7$.
(3) $b = 41$, $A = 87^\circ. 12'. 20''\cdot 3$, $B = 2^\circ. 47'. 39''\cdot 7$.
(4) $b = 527$, $A = 32^\circ. 31'. 13''\cdot 5$, $B = 57^\circ. 28'. 46''\cdot 5$.
(5) $b = 141$, $A = 82^\circ. 41'. 44''$, $B = 7^\circ. 18'. 6''$.
(6) $a = 748$, $A = 75^\circ. 23'. 18''\cdot 5$, $B = 14^\circ. 36'. 41''\cdot 5$.

(7) $a=736$, $A=69^\circ.\,38'.\,56''{\cdot}3$, $B=20^\circ.\,21'.\,3''{\cdot}7$.

(8) $a=200$, $A=18^\circ.\,10'.\,50''$, $B=71^\circ.\,49'.\,10''$.

(9) $c=565$, $A=29^\circ.\,14'.\,30''{\cdot}3$, $B=60^\circ.\,45'.\,29''{\cdot}7$.

(10) $c=565$, $A=44^\circ.\,29'.\,53''$, $B=45^\circ.\,30'.\,7''$.

LI. (Page 160.)

(1) 231·835 feet. (2) 93·97 feet.

(3) 36·6...feet; 70·7...feet; 100 feet. (4) 196 feet nearly.

(5) 460 yds. nearly. (6) 63°. 26′. 6″. (7) 88 yds. nearly.

(8) 33°. 23′. 54″·6. (6) 39°. 18′. 27″·5. (10) 104·93 feet.

(11) 45 feet. (12) 150 feet.

LII. (Page 167.)

(1) 67°. 22′. 48″·5. (2) 43°. 36′. 10″·1.

(3) 112°. 36′. 11″·5. (4) 29°. 51′. 46″.

LIII. (Page 172.)

(1) $A=87^\circ.\,44'.\,50''$, $C=49^\circ.\,7'.\,10''$. (2) $b=79{\cdot}063$.

(3) $b=219{\cdot}37$. (4) 57°. 14′. 21″ or 122°. 45′. 39″.

(5) 222 feet nearly. (6) 1210 yds, 1037·5 yds.

(7) 4584 yds, 4596 yds. (8) 34′. 22″·64.

(9) 85·28 ft. or 28·14 ft. (10) 107 feet nearly.

LIV. (Page 174.)

(1) $c=5$, $A=53^\circ.\,7'.\,48''{\cdot}4$, $B=36^\circ.\,52'.\,11''{\cdot}6$.

(2) $a=48$, $A=41^\circ.\,6'.\,43''{\cdot}5$, $B=48^\circ.\,53'.\,16''{\cdot}5$.

(3) $c = 353$, $A = 50^\circ.\ 24'.\ 8''\cdot 1$, $B = 39^\circ.\ 35'.\ 51''\cdot 9$.

(4) $a = 40$, $A = 5^\circ.\ 43'.\ 29''\cdot 3$, $B = 84^\circ.\ 16'.\ 30''\cdot 7$.

(5) $a = 84$, $b = 437$, $B = 79^\circ.\ 7'.\ 9''\cdot 6$.

(6) $a = 460$, $b = 429$, $B = 43^\circ.\ 0'.\ 10''\cdot 3$.

(7) $a = 280$, $b = 351$, $A = 38^\circ.\ 34'.\ 48''\cdot 3$.

(8) $a = 180$, $b = 299$, $A = 31^\circ.\ 2'.\ 53''\cdot 6$.

(9) $b = 231$, $c = 569$, $B = 23^\circ.\ 57'.\ 8''\cdot 1$.

(10) $a = 480$, $c = 481$, $B = 3^\circ.\ 41'.\ 42''\cdot 8$.

LV. (Page 174.)

(1) $A = 31^\circ.53'.26''\cdot 8$, $B = 8^\circ.\ 10'.\ 16''\cdot 4$, $C = 139^\circ.\ 56'.\ 16''\cdot 8$.

(2) $A = 84^\circ.32'.50''\cdot 5$, $B = 25^\circ.36'.30''\cdot 7$, $C = 69^\circ.\ 50'.\ 38''\cdot 8$.

(3) $A = 76^\circ.\ 18'.\ 52''$, $B = 35^\circ.\ 18'.\ 0''\cdot 9$, $C = 68^\circ.\ 23'.\ 7''\cdot 1$.

(4) $A = 62^\circ.\ 51'.\ 32''\cdot 9$, $B = 44^\circ.\ 29'.\ 53''$, $C = 72^\circ.\ 22'.\ 34''\cdot 1$.

(5) $A = 150^\circ.\ 8'.\ 14''$, $B = 17^\circ.\ 3'.\ 41''\cdot 5$, $C = 12^\circ.\ 48'.\ 4''\cdot 5$.

(6) $a = 101$, $c = 120$, $B = 11^\circ.\ 25'.\ 16''\cdot 3$.

(7) $a = 221$, $c = 222$, $B = 39^\circ.\ 18'.\ 27''\cdot 5$.

(8) $c = 78$, $A = 136^\circ.\ 23'.\ 49''\cdot 9$, $B = 11^\circ.\ 25'.16''\cdot 3$.

(9) $c = 408$, $A = 77^\circ.\ 19'.\ 10''\cdot 6$, $B = 5^\circ.\ 43'.\ 20''\cdot 3$.

(10) $c = 120$, $A = 110^\circ.\ 0'.\ 57''\cdot 5$, $B = 39^\circ.\ 18'.\ 27''\cdot 5$.

(11) $c = 102$, $A = 79^\circ.\ 36'.\ 40''$, $B = 33^\circ.\ 23'.\ 54''\cdot 6$.

(12) $c = 450$, $A = 81^\circ.\ 12'.\ 9''\cdot 3$, $B = 10^\circ.\ 52'.\ 50''\cdot 4$.

(13) $c = 312$, $A = 33^\circ.\ 23'.\ 54''\cdot 6$, $B = 15^\circ.\ 11'.\ 21''\cdot 4$.

(14) $c = 328$, $A = 45^\circ.\ 14'.\ 23''$, $B = 29^\circ.\ 51'.\ 46''$.

(15) $c=90$, $A=134^\circ.\ 45'.\ 37''$, $B=29^\circ.\ 51'.\ 46''$.

(16) $B=67^\circ.\ 22'.\ 48''\cdot5$ or $112^\circ.\ 37'.\ 11''\cdot5$.

(17) $B=62^\circ.\ 51'.\ 32''\cdot9$ or $117^\circ.\ 8'.\ 27''\cdot1$.

(18) $B=78^\circ.\ 22'.\ 32''\cdot3$ or $101^\circ.\ 37'.\ 27''\cdot7$.

(19) $B=75^\circ.\ 18'.\ 28''\cdot1$ or $104^\circ.\ 41'.\ 31''\cdot9$.

(20) $B=53^\circ.\ 26'.\ 0''\cdot6$.

LVI. (Page 184.)

1. $30\sqrt{3}$ sq. in.
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